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Exact and approximate controllability for distributed parameter systems

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This is the first part of an article which will be continued in the next volume of *Acta Numerica*. References in the text to Sections 2 to 7 refer to the continuation.

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The contents of the rest of this article (to appear in *Acta Numerica 1995*) are as follows:

0. INTRODUCTION

0.1. What is it all about

We consider a system whose *state* is given by the solution y to a Partial Differential Equation (PDE) of evolution, and which contains *control functions*, denoted by v.

Let us write in a formal fashion for the time being. The state equation is written

$$\frac{\partial y}{\partial t} + \mathcal{A}(y) = \mathcal{B}v, \tag{0.1}$$

where y is a scalar or a vector valued function.

In (0.1), $\mathcal{A}$ is a set of Partial Differential Operators (PDO), linear or nonlinear (at least for the time being). In (0.1), v denotes the control and $\mathcal{B}$ maps the 'space of controls' into the state space. It goes without saying that all this has to be made precise. This will be the task of the following sections.

The PDE (0.1) should include *boundary conditions.* We do not make them explicit here. They are supposed to be contained in the abstract formulation (0.1), where v can be applied inside the domain $\Omega \subset \mathbb{R}^d$ where (0.1) is considered (one says that v is a *distributed* control) or on the boundary Γ of Ω (or a part of it). One says then that v is *a boundary control.*

If v is applied at points of Ω, v is said to be a *pointwise* control.

One has also to add *initial conditions* to (0.1): if we assume that $t = 0$ is the initial time, then these initial conditions are given by

$$y|_{t=0} = y_0, \tag{0.2}$$

y_0 being a given element of the state space.

It will be assumed that, given v (in a suitable space), problem (0.1), (0.2) (and the boundary conditions included in formulation (0.1)) *uniquely defines a solution.*

It is a function (scalar or vector valued) of $x \in \Omega$, $t > 0$, and of y_0 and v. *We shall denote by $y(v)(= \{x, t\} \rightarrow y(x, t; v))$ this solution.*

We shall denote by $y(t; v)$ the function $x \rightarrow y(x, t; v)$.

Then (0.2) can be written as

$$y(0; v) = y_0. \tag*{(0.2)$_*$}$$

Remark 0.1 The notions introduced below can be extended to situations where the *uniqueness* of the solution to (0.1), (0.2) is *not known.* We are thinking here of the Navier–Stokes equations in $\Omega \subset \mathbb{R}^d$, $d = 3$, and the equations related to it.

We can now introduce the notion of *controllability*, either exact or approximate.

Let $T > 0$ *be given* and let y_T be a given element of the state space. We want to 'drive the system' from y_0 at $t = 0$ to y_T at $t = T$, i.e. we want to find v such that

$$y(T; v) = y_T. \tag{0.3}$$

If this is possible for every y_T in the state space, one says that the system is *controllable* (or *exactly controllable*).

If – as we shall see in most of the examples – condition (0.3) is too strict, it is natural to replace it by

$$y(T; v) \text{ belongs to a 'small' neighbourhood of } y_T. \tag{0.4}$$

If this is possible, one says that the system is *approximately controllable.* Otherwise the system is *not* controllable.

Before starting with precise examples, we want to say a few words concerning the motivation for studying these problems.

0.2. Motivation

There are several aspects which make controllability problems important in practice.

Aspect 1 At a *given time horizon* we want the system under study to behave *exactly* as we wish (or in a manner arbitrarily close to it).

Problems of this type are common in science and engineering: we would like, for example, to have the temperature (or pressure) of a system equal or very close to a given value – globally or locally – at a given time. *Chemical engineering* is an important source of such problems, a typical example in that direction being the design of car catalytic converters; in this example chemical reactions have to take place leading to the 'destruction' at a given time horizon (very small in practice) of the polluting chemicals contained in the exhaust gases (the modelling and numerical simulation of catalytic converter systems are discussed in Engquist, Gustafsson and Vreeburg (1978), Friedman (1988, Ch. 7); see also Friend (1993)).

Aspect 2 For *linear* systems, it is known (cf. Russell (1978)) that exact controllability is equivalent to the possibility of *stabilizing* the system.

Stabilization problems abound, in particular in (large) composite structures, the so called 'multi-body systems' made of many different parts, which can be considered as three-, two- or one-dimensional and which are 'tied' together by *junctions* and *joints*. The modelling and analysis of such systems is the subject of many interesting studies. We wish to mention here Ph. Ciarlet and his collaborators (see, for example, Ciarlet (1990a,b) and Ciarlet, Le Dret and Nzengwa (1989)), Hubert and Palencia (1989), Lagnese and Leugering (1994), Simo and his collaborators (see for example, Laursen and Simo (1994)), Park and his collaborators (see for example Park, Chiou and Downer (1990) and Downer, Park and Chiou (1992)).

Studying controllability is *one* approach to stabilization (Lions, 1988a).

Aspect 3 Controllability and reversibility. Suppose we have a system which *was* in a state z_1 at time $-t_0$, $t_0 > 0$, and which is *now* in the state y_0.

We would like to have the system *returning* to a state as close as possible to z_1, i.e. $y_T = z_1$. If this is possible, it means some kind of 'reversibility'. What we have in mind here are *environment systems*; should they be 'local' or 'global' in the space variables?

Noncontrollable subsystems can suffer 'irreversible' changes (cf. Lions (1990) and Diaz (1991)).

We return now to the general questions of Section 0.1, making them more precise before giving examples.

0.3. Topologies and numerical methods

The topology of the state space appears explicitly in condition (0.4). It is obvious that approximate controllability *depends* on the choice of the topology on the state space, i.e. of the state space, itself. Actually *exact* controllability depends on the choice of the state space as well.

The choice of the state space is therefore an obviously fundamental issue for the *theory*.

We want to emphasize that it is *also* a fundamental issue from the *numerical point of view*.

If one has (as we shall see in several situations) exact or approximate controllability in a very general space (which can include elements which are not distributions but 'ultra distributions') and *not* in a classical space of smooth (or sufficiently smooth) functions, then the numerical approximation will *necessarily* develop singularities and 'remedies' should be based on knowledge of the topology where theoretical convergence takes place. We shall return to these issues in the following sections; some of them have been addressed in, e.g., Dean, Glowinski and Li (1989), Glowinski, Li and Lions (1990), Glowinski and Li (1990), Glowinski 1992a), where various filtering techniques are discussed in order to eliminate the numerical singularities mentioned above.

In the following section we shall address the question, also very general in nature, namely: *how to choose the control*?

0.4. Choice of control

Let us return to the general formulation (0.1), (0.2), (0.3) (or (0.4)). If there exists *one* control v achieving these conditions, then there exist in general *infinitely many other* vs also achieving these conditions. Which one should we choose and how?

A most important question is: how to choose the *norm* (we are always working in Banach or Hilbert spaces) for the vs? This is related to the *topology* of the state space. It is indeed clear that the regularity (or irregularity!) properties of v and of y in (0.1) are related. Let us assume that a norm $|||v|||$ is chosen.

Once this choice is made, a natural formulation of the problem is then to find

$$\inf |||v|||, \tag{0.5}$$

among all vs such that (0.1), (0.2), (0.3) (or (0.4)) take place.

Remark 0.2 There is still some flexibility here, since problem (0.5) makes sense if one replaces $|||\cdot|||$ by a *stronger* norm. This remark may be of *practical* interest, as we shall see later on.

Remark 0.3 One can meet questions of controllability for systems depending on 'small' parameters. Two classical (by now) examples are:

(i) *singular perturbations*,

(ii) *homogenization* which is important for the controllability of structures made of *composite materials*.

In these situations one has to introduce either *families* of norms in (0.5) or norms *equivalent* to $|||\cdot|||$ but which depend on the homogenization parameter.

0.5. Relaxation of the controllability notion

Let us return again to (0.1), (0.2).

Condition (0.3) concerns the state y itself. In a 'complex system' this condition can be (and will be in general) unnecessarily strong.

We may want some *subsystem* to behave according to our wishes. We may also want *average* values to behave accordingly, etc.

A general formulation is as follows.

We consider an operator

$$C \in \mathcal{L}(Y, \mathcal{H}), \tag{0.6}$$

where Y is the state space (chosen!) and where $\mathcal{H}$ is another Banach or Hilbert space (the *observation* space). Think, for instance, of C as being an *averaging* operator.

Then we 'relax' (0.3) (respectively (0.4)) as follows

$$Cy(T;v) = h_T, \quad h_T \text{ given in } \mathcal{H} \tag{0.7}$$

(respectively

$$Cy(T;v) \text{ belongs to some neighbourhood of } h_T \text{ in } \mathcal{H}). \tag{0.8}$$

Then we consider (0.5) where v is subject to (0.7) (respectively (0.8)).

0.6. Various remarks

Remark 0.4 For most examples considered in this article, the control function is either distributed (or pointwise) or of a boundary nature. It can also be a *geometrical* one. Namely we can consider the domain Ω as variable or, to be more precise, at least a part of the boundary of Ω is variable, and we want to 'move this part of the boundary' in order to drive the system from a given state to another one. In summary we look for *controllability by a suitable variable geometry.* Problems of this type are discussed in Bushnell and Hefner (1990); they mostly concern *drag reduction* for viscous flow (see also Sellin and Moses (1989)). We shall return to this on other occasions.

Remark 0.5 Some recent events have shown the importance of *stealth technologies.* The related problems are very complicated from the modelling, mathematical, numerical and engineering points of view; several approaches can be envisaged (they do not exclude one another) such as active control, passive control through well chosen coating materials and/or well chosen shape, use of decoy strategies, etc. Indeed these methods can be applied for planes and submarines as well. These problems justify a book in themselves and will not be specifically addressed here. We think, however, that various notions related to controllability including the recently introduced concept of *sentinels* can be most helpful in the formulation and solution of stealth problems. It is also worth mentioning that the *exact controllability* based solution methods for the *Helmholtz equation at large wave numbers*, described in Section 6.13, have been motivated by stealth issues.

1. DISTRIBUTED AND POINTWISE CONTROL FOR LINEAR DIFFUSION EQUATIONS

1.1. First example

Let Ω be a bounded open set in $\mathbb{R}^d$ ($d \leq 3$ in the applications).

Remark 1.1 The 'boundedness' hypothesis is by no means a strict necessity.

We shall also assume that $\Gamma = \partial\Omega$ is 'sufficiently smooth', which is also not mandatory.

Let $\mathcal{O} \subset \Omega$ be an open subset of Ω.

Remark 1.2 We emphasize here at the very beginning that $\mathcal{O}$ can be *arbitrarily 'small'.*

The control function v will be with support in $\bar{\mathcal{O}}$. It is a *distributed control.*

The state equation is given by

$$\frac{\partial y}{\partial t} + Ay = v\chi_{\mathcal{O}} \text{ in } \Omega \times (0,T), \tag{1.1}$$

where $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$ and where A is a *second-order elliptic operator*, with variable coefficients. *The coefficients of A can also depend on t.*

Example 1.1 A typical elliptic operator A is the one defined by

$$Ay = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \sum_{j=1}^{d} a_{ij} \frac{\partial y}{\partial x_j} + \mathbf{V}_0 \cdot \boldsymbol{\nabla} y, \tag{1.2}$$

where, in (1.2), $\boldsymbol{\nabla} = \{\partial/\partial x_i\}_{i=1}^d$ and where

(i) The coefficients a_{ij} belong to $L^\infty(\Omega)$ $\forall i,j, 1 \leq i,j \leq d$, and the matrix function $(a_{ij})_{1\leq i,j \leq d}$ satisfies

$$\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \alpha \|\boldsymbol{\xi}\|^2 \ \forall \boldsymbol{\xi} = \{\xi_i\}_{i=1}^d \in \mathbb{R}^d, \text{ a.e. in } \Omega, \tag{1.3}$$

with $\alpha > 0$ and $\|\cdot\|$ the canonical Euclidean norm of $\mathbb{R}^d$.

(ii) The vector $\mathbf{V}_0$ is *divergence-free* (i.e. $\nabla \cdot \mathbf{V}_0 = 0$) and belongs to $(L^\infty(\Omega))^d$.

(iii) We have used the *dot product* notation for the canonical Euclidean scalar product of $\mathbb{R}^d$, i.e.

$$\boldsymbol{\eta} \cdot \boldsymbol{\xi} = \sum_{i=1}^{d} \eta_i \xi_i \ \forall \boldsymbol{\eta} = \{\eta_i\}_{i=1}^d, \ \boldsymbol{\xi} = \{\xi_i\}_{i=1}^d \in \mathbb{R}^d.$$

If the above hypothesis on the a_{ij}s and $\mathbf{V}_0$ are satisfied, then the *bilinear form* $a(\cdot\,,\cdot)$ defined by

$$a(y,z) = \sum_{i=1}^{d}\sum_{j=1}^{d} \int_\Omega a_{ij} \frac{\partial y}{\partial x_j}\frac{\partial z}{\partial x_i}\,\mathrm{d}x + \int_\Omega \mathbf{V}_0 \cdot \nabla y z\,\mathrm{d}x \tag{1.4}$$

is *continuous* over $H^1(\Omega) \times H^1(\Omega)$; it is also *strongly elliptic* over $H_0^1(\Omega) \times H_0^1(\Omega)$ since we have, from (1.3) and from $\nabla \cdot \mathbf{V}_0 = 0$, the following relation

$$a(y,y) \geq \alpha \int_\Omega |\nabla y|^2\,\mathrm{d}x \ \forall y \in H_0^1(\Omega). \tag{1.5}$$

If $\mathbf{V}_0 = \mathbf{0}$ and if $a_{ij} = a_{ji}$ $\forall i,j,\ 1 \leq i,\ j \leq d$, then the bilinear form $a(\cdot\,,\cdot)$ is *symmetric*.

Above, $H^1(\Omega)$ and $H_0^1(\Omega)$ are the *functional spaces* defined as follows

$$H^1(\Omega) = \{\varphi \mid \varphi \in L^2(\Omega),\ \partial\varphi/\partial x_i \in L^2(\Omega)\ \forall i = 1,\dots,d\}, \tag{1.6}$$

and

$$H_0^1(\Omega) = \{\varphi \mid \varphi \in H^1(\Omega),\ \varphi = 0 \text{ on } \Gamma\}, \tag{1.7}$$

respectively. Equipped with the classical *Sobolev norm*

$$\|\varphi\|_{H^1(\Omega)} = \left(\int_\Omega (\varphi^2 + |\nabla\varphi|^2)\,\mathrm{d}x\right)^{1/2},$$

and with the corresponding *scalar product*

$$(\varphi,\psi)_{H^1(\Omega)} = \int_\Omega (\varphi\psi + \nabla\varphi\cdot\nabla\psi)\,\mathrm{d}x,$$

$H^1(\Omega)$ and $H_0^1(\Omega)$ are *Hilbert spaces.*

Since Ω is bounded,

$$\varphi \to \left(\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x\right)^{1/2}$$

defines a norm over $H_0^1(\Omega)$ which is *equivalent* to the above $H^1(\Omega)$ norm, the corresponding scalar product being

$$\{\varphi,\psi\} \to \int_\Omega \nabla\varphi\cdot\nabla\psi\,\mathrm{d}x.$$

If we denote by $H^{-1}(\Omega)$ the *dual space* of $H_0^1(\Omega)$, then the above operator A is *linear* and *continuous* from $H^1(\Omega)$ into $H^{-1}(\Omega)$ and is an *ismorphism* from $H_0^1(\Omega)$ *onto* $H^{-1}(\Omega)$. □

Back to (1.1), and motivated by the class of elliptic operators discussed in the above example, we shall suppose from now on that operator A is *linear and continuous* from $H^1(\Omega)$ into $H^{-1}(\Omega)$ and that it satisfies the following (*ellipticity*) property

$$\langle A\varphi, \varphi\rangle \geq \alpha\|\varphi\|^2_{H^1(\Omega)} \ \forall\varphi \in H_0^1(\Omega),$$

where, in the above relation, α is a *strictly positive* constant and where $\langle\cdot, \cdot\rangle$ denotes the *duality pairing* between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Operator A is *symmetric* over $H_0^1(\Omega)$ if

$$\langle A\varphi, \psi\rangle = \langle A\psi, \varphi\rangle \ \forall\varphi, \psi \in H_0^1(\Omega).$$

The *bilinear* form

$$\{\varphi, \psi\} \to \langle A\varphi, \psi\rangle : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

will be denoted by $a(\cdot, \cdot)$ and is symmetric if and only if A is self-adjoint.

In order to fix ideas and to make things as simple as possible, we add to (1.1) the following boundary condition, of *Dirichlet type*,

$$y = 0 \text{ on } \Sigma = \Gamma \times (0, T). \tag{1.8}$$

The initial condition is

$$y(0) = y_0, \tag{1.9}$$

where y_0 is given in $L^2(\Omega)$.

We shall assume that

$$v \in L^2(\mathcal{O} \times (0, T)). \tag{1.10}$$

We emphasize that this is a *choice* which is by no means compulsory. We shall return to this. We begin with (1.10) since it is the *simplest* possible choice, at least from a theoretical point of view.

It is a well known fact (cf. for instance Lions (1961), Lions and Magenes (1968)) that (1.1), (1.8), (1.9) admits a unique solution (denoted sometimes as $t \to y(t; v)$, with $y(t; v) = x \to y(x, t; v)$) which has the following properties

$$y \in L^2(0, T; H_0^1(\Omega)), \partial y/\partial t \in L^2(0, T; H^{-1}(\Omega)), \tag{1.11}$$

$$y \text{ is continuous from } [0, T] \to L^2(\Omega). \tag{1.12}$$

We are going to study the *(approximate) controllability* of problem (1.1), (1.8), (1.9).

1.2. Approximate controllability

As a preliminary remark, we note that *exact* controllability is going to be difficult. Indeed, if we assume that the coefficients of A are smooth (respectively real analytic) then the solution y is, at time $T > 0$, *smooth outside* $\mathcal{O}$ (respectively *real analytic outside* $\mathcal{O}$).

Therefore if y_T is given in $L^2(\Omega)$ – which is a natural choice if we take (1.12) into account – the condition of exact controllability

$$y(T) = y_T$$

will be, in general, *impossible.*

This will become more precise below. For the time being, we start with the *approximate controllability.* In that direction a key result is given by the following

Proposition 1.1 When v spans $L^2(\mathcal{O} \times (0,T)), y(T;v)$ spans an affine subspace which is dense in $L^2(\Omega)$.

Proof.

(i) Let Y_0 be the solution to (1.1), (1.8), (1.9) for $v = 0$. Then $y(T;v) - Y_0(T)$ describes a subspace of $L^2(\Omega)$ and we have to show the *density* of this subspace. It amounts to proving the above density result assuming $y_0 = 0$.

(ii) We then apply the *Hahn–Banach theorem*, as in Lions (1968) (so that the present proof is *not constructive*).

Let us consider indeed an element $f \in L^2(\Omega)$ such that

$$(y(T;v), f)_{L^2(\Omega)} = 0 \ \forall v \in L^2(\mathcal{O} \times (0,T)). \tag{1.13}$$

We introduce ψ as the solution to

$$-\frac{\partial \psi}{\partial t} + A^*\psi = 0 \text{ in } \Omega \times (0,T), \tag{1.14}$$

where A^* is the adjoint operator of A and where ψ also satisfies

$$\psi = 0 \text{ on } \Sigma, \tag{1.15}$$

$$\psi(x,T) = f(x). \tag{1.16}$$

Then multiplying (1.14) by $y(v)$ and applying *Green's formula*, we obtain

$$(y(T;v), f)_{L^2(\Omega)} = \iint_{\mathcal{O}\times(0,T)} \psi v \, \mathrm{d}x \, \mathrm{d}t. \tag{1.17}$$

Therefore (1.13) is equivalent to

$$\psi = 0 \text{ in } \mathcal{O} \times (0,T). \tag{1.18}$$

It then follows from the Mizohata (1958) uniqueness theorem, that

$$\psi \equiv 0 \text{ in } \Omega \times (0,T) \tag{1.19}$$

so that $f = 0$, which proves the proposition. □

Remark 1.3 Mizohata's theorem supposes that the coefficients of A are sufficiently smooth (cf. also Saut and Scheurer (1987)).

Remark 1.4 A similar density property holds true if v spans, say, the space of those functions which are C^∞ and with compact support in $\mathcal{O} \times (0,T)$. This fact gives a lot of flexibility to the formulation which follows.

Remark 1.5 Suppose we would like to drive the system at time T 'close' to a state y_T *containing some singularities.* To fix ideas (but there is also much flexibility here) suppose that

$$y_T \in H^{-1}(\Omega). \tag{1.20}$$

Then it *may* be sensible to admit fairly *general controls*, such as

$$v \in L^2(0,T;H^{-1}(\mathcal{O})) \tag{1.21}$$

or even more general ones. We shall not pursue these lines here.

1.3. Formulation of the approximate controllability problem

As we have seen in Section 1.2, we do not restrict the generality by assuming that $y_0 = 0$ (it amounts to replacing y_T by $y_T - Y_0(T)$).

Let B be the unit ball of $L^2(\Omega)$. We want

$$y(T;v) \text{ to belong to } y_T + \beta B, \beta > 0 \text{ (arbitrarily small).} \tag{1.22}$$

According to Proposition 1.1 there are controls vs (actually infinitely many such vs) such that (1.22) holds true. Among all these vs, we want to find those which are solutions to the following minimization problem:

$$\inf_v \tfrac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \,\mathrm{d}x\,\mathrm{d}t, \quad v \in L^2(\mathcal{O} \times (0,T)),\ y(T;v) \in y_T + \beta B. \tag{1.23}$$

In fact problem (1.23) admits a unique solution. We want to construct numerical approximation schemes to find it.

Before we proceed, a few remarks are now in order.

Remark 1.6 All that is stated above is true with

$$\begin{aligned} &T > 0 && \text{arbitrarily small,}\\ &\mathcal{O} \subset \Omega && \text{arbitrarily 'small',}\\ &\beta > 0 && \text{also arbitrarily small.}\end{aligned}$$

Letting $\beta \to 0$ *will be, in general, impossible.* This will be made explicit below.

Remark 1.7 Choices other than (1.23) are possible. The 'obvious' candidates are

$$\inf_v \|v\|_{L^1(\mathcal{O}\times(0,T))},\ v \in L^1(\mathcal{O} \times (0,T)),\ y(T;v) \in y_T + \beta B, \tag{1.24}$$

or

$$\inf_v \|v\|_{L^\infty(\mathcal{O}\times(0,T))},\quad v \in L^\infty(\mathcal{O} \times (0,T)),\quad y(T;v) \in y_T + \beta B. \tag{1.25}$$

Other – more subtle – choices may be of interest. We shall return to this below.

1.4. Dual problem

We are going to apply the *Duality Theory of Convex Analysis* to problem (1.23).

We define the following functionals and operator

$$F_1(v) = \tfrac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \,dx\,dt, \tag{1.26}$$

$$F_2(f) = \begin{cases} 0 & \text{for } f \text{ in } L^2(\Omega),\quad f \in y_T + \beta B \\ +\infty & \text{otherwise,} \end{cases} \tag{1.27}$$

(F_2 is a 'proper' convex functional)

$$Lv = y(T;v), \tag{1.28}$$

so that

$$L \in \mathcal{L}(L^2(\mathcal{O} \times (0,T)); L^2(\Omega)). \tag{1.29}$$

Then problem (1.23) where the infimum is taken over all vs satisfying (1.22) is *equivalent* to the following minimization problem

$$\inf_{v\in L^2(\mathcal{O}\times(0,T))} [F_1(v) + F_2(Lv)]. \tag{1.30}$$

We can now apply the duality theorem of W. Fenchel and T.R. Rockafellar (cf. Ekeland and Temam (1974)). It gives

$$\inf_{v\in L^2(\mathcal{O}\times(0,T))} [F_1(v) + F_2(Lv)] = - \inf_{f\in L^2(\Omega)} [F_1^*(L^*f) + F_2^*(-f)] \tag{1.31}$$

where F_i^* is the conjugate function of F_i and L^* is the adjoint operator of L.

We have

$$\begin{aligned} F_1^*(v) &= \sup_{\hat{v}\in L^2(\mathcal{O}\times(0,T))} [(v,\hat{v}) - F_1(\hat{v})] = F_1(v), \\ F_2^*(f) &= \sup_{\hat{f}\in y_T+\beta B} (f,\hat{f}) = (f, y_T) + \beta\|f\|, \end{aligned}$$

where $\|f\|$ = norm of f in $L^2(\Omega)$ and where (f, y_T) = scalar product of f

and y_T in $L^2(\Omega)$. We now compute L^*. Given f in $L^2(\Omega)$, we define ψ as the solution to (1.14)–(1.16).

Then, one verifies easily (actually one uses (1.17)) that

$$L^* f = \psi \chi_{\mathcal{O}}, \qquad \chi_{\mathcal{O}} = \text{characteristic function of } \mathcal{O}. \tag{1.32}$$

Therefore (1.31) *gives*

$$\inf_{v \in L^2(\mathcal{O}\times(0,T))} [F_1(v) + F_2(Lv)]$$
$$= - \inf_{\hat{f} \in L^2(\Omega)} \left[\frac{1}{2} \iint_{\mathcal{O}\times(0,T)} \hat{\psi}^2 \, dx \, dt - (\hat{f}, y_T) + \beta \|\hat{f}\| \right], \tag{1.33}$$

where $\hat{\psi}$ *is the solution to*

$$-\frac{\partial \hat{\psi}}{\partial t} + A^* \hat{\psi} = 0 \text{ in } \Omega \times (0,T), \quad \hat{\psi}(T) = \hat{f}, \quad \hat{\psi} = 0 \text{ on } \Sigma. \tag{1.34}$$

Minimizing the functional on the right-hand side of (1.33), *where the state function is now given by* (1.34), *is the dual problem.*

Remark 1.8 Problem (1.33), (1.34) admits a unique solution. Let f denote this solution. Then *the solution* u *to problem* (1.23) *is given by*

$$u = \psi \chi_{\mathcal{O}}, \tag{1.35}$$

where ψ is the solution to (1.34) corresponding to $\hat{f}$.

Remark 1.9 We now want to give constructive algorithms for finding the solution to the dual problem, hence for the solution to the primal problem (using (1.35)).

Remark 1.10 As is classical in questions of this sort, relation (1.33) leads to lower and upper bounds, hence to some error estimates.

1.5. Direct solution to the dual problem

Given f in $L^2(\Omega)$, let us set

$$[f] = \|\psi\|_{L^2(\mathcal{O}\times(0,T))}. \tag{1.36}$$

We observe that $[f]$ is a norm on $L^2(\Omega)$. Indeed, if $[f] = 0$ then $\psi = 0$ in $\mathcal{O} \times (0,T)$, hence (according to the proof of Proposition 1.1) $f = 0$ follows.

Let us now introduce a *variational inequality* expressing that f realizes the minimum on the right-hand side of (1.33).

It is given by

$$\iint_{\mathcal{O}\times(0,T)} \psi(\hat{\psi} - \psi) \, dx \, dt - (y_T, \hat{f} - f) + \beta \|\hat{f}\| - \beta \|f\| \geq 0 \; \forall \hat{f} \in L^2(\Omega), \tag{1.37}$$

where $\hat{\psi}$ is the solution to (1.34) corresponding to $\hat{f}$.

Using (1.36), this is equivalent to

$$[f,\hat{f}-f]-(y_T,\hat{f}-f)+\beta\|\hat{f}\|-\beta\|f\|\geq 0\ \forall\hat{f}\in L^2(\Omega). \tag{1.38}$$

Let us introduce the 'adjoint' state function y defined by

$$\frac{\partial y}{\partial t}+Ay=\psi\chi_{\mathcal{O}} \text{ in } \Omega\times(0,T),\quad y(0)=0,\quad y=0 \text{ on } \Sigma. \tag{1.39}$$

Multiplying the first equation in (1.39) by $\hat{\psi}-\psi$ gives

$$\iint_{\mathcal{O}\times(0,T)}\psi(\hat{\psi}-\psi)\,\mathrm{d}x\,\mathrm{d}t=(y(T),\hat{f}-f). \tag{1.40}$$

Let us set

$$y(T)=y(T;f)=\Lambda f, \tag{1.41}$$

where, given f, one computes ψ by (1.34) and then y by (1.39).

Then (1.37) (or (1.38)) can be written

$$(\Lambda f,\hat{f}-f)-(y_T,\hat{f}-f)+\beta\|\hat{f}\|-\beta\|f\|\geq 0\ \forall\hat{f}\in L^2(\Omega). \tag{1.42}$$

Remark 1.11 The equivalence between problems (1.38) and (1.42) relies on the following relation

$$[f,\hat{f}]=(\Lambda f,\hat{f})\forall f,\hat{f}\in L^2(\Omega). \tag{1.43}$$

Remark 1.12 Operator Λ satisfies

$$\Lambda\in\mathcal{L}(L^2(\Omega);L^2(\Omega)),\quad \Lambda=\Lambda^*,\ \Lambda\geq 0. \tag{1.44}$$

It follows from (1.44) that the (unique) solution to problem (1.42) is also the solution to

$$\inf_{\hat{f}\in L^2(\Omega)}\left[\tfrac{1}{2}(\Lambda\hat{f},\hat{f})-(y_T,\hat{f})+\beta\|\hat{f}\|\right]. \tag{1.45}$$

We can summarize by the following

Theorem 1.1 (i) *We have the identity*

$$\inf_{\substack{v\\ y(T;v)\in y_T+\beta B}}\tfrac{1}{2}\iint_{\mathcal{O}\times(0,T)}v^2\,\mathrm{d}x\,\mathrm{d}t$$
$$=-\inf_{\hat{f}\in L^2(\Omega)}\left[\tfrac{1}{2}\iint_{\mathcal{O}\times(0,T)}\hat{\psi}^2\,\mathrm{d}x\,\mathrm{d}t-(y_T,\hat{f})+\beta\|\hat{f}\|\right], \tag{1.46}$$

where $\hat{\psi}$ *is given by* (1.34).

(ii) *The unique solution* f *of the dual problem is the solution of* (1.45) *where* Λ *is defined by* (1.41), *i.e.* $\Lambda f=y(T)$ *where*

$$-\frac{\partial\psi}{\partial t}+A^*\psi=0 \text{ in } \Omega\times(0,T),\quad \psi(T)=f,\ \psi=0 \text{ on } \Sigma \qquad (1.47)_1$$

$$\frac{\partial y}{\partial t} + Ay = \psi\chi_{\mathcal{O}} \text{ in } \Omega \times (0,T), \quad y(0) = 0, y = 0 \text{ on } \Sigma. \tag{1.47$_2$}$$

(iii) *The unique solution* u *of* (1.46) *is given by*

$$u = \psi\chi_{\mathcal{O}}. \tag{1.48}$$

Application As a corollary – *which we have to make precise!* – one obtains the general principle of a solution method, namely

(i) Guess the solution f of problem (1.46).
(ii) Compute the corresponding value of ψ.
(iii) Use an iterative method to compute the inf in f, using the right-hand side of (1.46) or using (1.45).

This will be the task of Section 1.8. Before that several remarks have to be made.

Remark 1.13 The optimal control v – with respect to the *choice* of

$$\tfrac{1}{2}\iint_{\mathcal{O}\times(0,T)} v^2 \,\mathrm{d}x\,\mathrm{d}t$$

as the quantity to minimize – is given by (1.48), where ψ is the solution of the parabolic equation $(1.47)_1$. Therefore ψ is *smooth* (the smoother the coefficients of A, the smoother ψ will be). *In other words,* u *is smooth.*

This remark *excludes* the possibility of finding an optimal control of the 'bang–bang' type.

Of course trying to find an optimal control satisfying some kind of bang–bang principle is by no means compulsory! But knowing in advance that such a property holds true may be of some help.

The first idea which comes to mind is to replace

$$\|v\|_{L^2(\mathcal{O}\times(0,T))} \text{ by } \|v\|_{L^\infty(\mathcal{O}\times(0,T))};$$

this possibility will be discussed in Section 1.7.

Remark 1.14 (Further comments on exact controllability.) Exact controllability corresponds to $\beta = 0$ in (1.45), or, equivalently, to

$$\inf_{\hat{f}}[\tfrac{1}{2}[\hat{f}]^2 - (y_T, \hat{f})], \hat{f} \in L^2(\Omega). \tag{1.49}$$

Let us denote by $\widehat{L^2(\Omega)}$ the completion of $L^2(\Omega)$ for the norm $[\hat{f}]$. Due to the smoothness properties of parabolic equations, $\widehat{L^2(\Omega)}$ will contain (except for the case, without practical interest, where $\mathcal{O} = \Omega$) very singular distributions and even distributions of infinite order (outside $\bar{\mathcal{O}}$), i.e. ultra-distributions.

Then (1.49) admits a unique solution f_0 iff

$$y_T \in (\widehat{L^2(\Omega)})' \tag{1.50}$$

the dual of $\widehat{L^2(\Omega)}$, when $L^2(\Omega)$ is identified with its dual. It means that exact controllability is possible iff y_T belongs to a '*very small*' space, namely $(\widehat{L^2(\Omega)})'$.

We also have the following convergence result: let f_β be the unique solution to (1.46), *then* $f_\beta \to f_0$ *in* $L^2(\Omega)$ *as* $\beta \to 0$ *iff* $y_T \in (\widehat{L^2(\Omega)})'$.

Remark 1.15 Another way of expressing this is to observe that Λ is an *isomorphism* from $\widehat{L^2(\Omega)}$ onto its dual. This is closely related to the *Hilbert Uniqueness Method* (HUM) as introduced in Lions (1988a,b).

1.6. Penalty arguments

In problems where there are many constraints of a different nature, penalty arguments can be used in a very large number of ways.

In the present section we are going to 'penalize' the constraint

$$y(T; v) \text{ belongs to } y_T + \beta B. \tag{1.51}$$

This can also be done in many ways!

One possibility is to introduce a *smooth* functional over $L^2(\Omega)$ which is zero on the ball $y_T + \beta B$, and > 0 outside the ball; let $\mu(\cdot)$ be such a functional. Then one can consider

$$\inf_v \left[\tfrac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \, dx \, dt + k\mu(y(T; v))\right],$$
$$v \in L^2(\mathcal{O} \times (0,T)), \quad k > 0 \text{ 'large'}. \tag{1.52}$$

Another possibility is the following. One introduces

$$J_k(v) = \tfrac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \, dx \, dt + \frac{k}{2}\|y(T; v) - y_T\|^2, \tag{1.53}$$

where $k > 0$ is 'large' and where $\|\cdot\|$ denotes the $L^2(\Omega)$ norm.

Then one considers the problem

$$\inf_v J_k(v), \quad v \in L^2(\mathcal{O} \times (0,T)). \tag{1.54}$$

This problem *admits a unique solution*, denoted by u_k. Let us verify the following result:

$$\left\{ \begin{array}{l} \text{There exists } k \text{ large enough such that the solution } u_k \text{ of (1.54)} \\ \text{satisfies } \|y(T; u_k) - y_T\| \le \beta. \end{array} \right. \tag{1.55}$$

Before proving (1.55) let us make the following remark.

Remark 1.16 It follows from (1.55) that u_k is, for k large enough, *one* control such that $y(T; u_k) \in y_T + \beta B$. Of course it has no reasons to coincide with the solution u_β of

$$\inf \tfrac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \,\mathrm{d}x\,\mathrm{d}t, \quad v \in L^2(\mathcal{O} \times (0,T)), \quad y(T; v) \in y_T + \beta B.$$

Remark 1.17 The proof to follow is *not* constructive, therefore it does not give a 'constructive choice' for k which is a difficulty since β 'disappears' in problem (1.54). *We make below a constructive proposal for the choice of k.*

Remark 1.18 Of course, given k, the *optimality system* for problem (1.54) is quite classical. One obtains

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = \psi\chi_{\mathcal{O}} \text{ in } \Omega \times (0,T), \ y(0) = 0, \ y = 0 \text{ on } \Sigma, \\ -\dfrac{\partial \psi}{\partial t} + A^*\psi = 0 \text{ in } \Omega \times (0,T), \ \psi(T) = k(y_T - y(T)), \ \psi = 0 \text{ on } \Sigma. \end{cases} \tag{1.56}$$

The optimal control u_k is given by $\psi\chi_{\mathcal{O}}$ where ψ is the solution obtained by solving (1.56).

It is worth noticing that if one denotes the function $\psi(T)$ by f, then f satisfies the functional equation

$$(k^{-1}\mathbf{I} + \Lambda)f = y_T, \tag{1.57}$$

where operator Λ is still defined by (1.41) (see Section 1.5).

Proof of (1.55). Given $\varepsilon > 0$, there exists a control w such that

$$\|y(T; w) - y_T\| \leq \varepsilon. \tag{1.58}$$

This follows from the approximate controllability result and it is not constructive.

Then

$$J_k(u_k) \leq \tfrac{1}{2} \iint_{\mathcal{O}\times(0,T)} w^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{k\varepsilon^2}{2}, \tag{1.59}$$

so that

$$\|y(T; u_k) - y_T\|^2 \leq \frac{1}{k} \iint_{\mathcal{O}\times(0,T)} w^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2. \tag{1.60}$$

We choose $\varepsilon = \beta/\sqrt{2}$, then w is chosen so that (1.58) holds and we choose k such that

$$\frac{1}{k} \iint_{\mathcal{O}\times(0,T)} w^2 \,\mathrm{d}x\,\mathrm{d}t \leq \tfrac{1}{2}\beta^2.$$

Then (1.60) implies (1.55). □

Remark 1.19 In general (i.e. for y_T generically given in $L^2(\Omega)$) the above process does *not* converge as $k \to +\infty$ (otherwise it would give exact controllability at the limit!).

Remark 1.20 It will remain to solve (1.56) *if we have a way to choose k.* This is what we propose now.

Duality on $J_k(v)$:
We introduce

$$F_1(v) = \tfrac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \, \mathrm{d}x \, \mathrm{d}t, \quad F_2(f) = \tfrac{1}{2} k \|f - y_T\|^2, \ Lv = y(T; v). \tag{1.61}$$

We have

$$\inf_v J_k(v) = \inf_v (F_1(v) + F_2(Lv))$$

and using *duality* as in previous sections (and with similar notation), we obtain

$$\inf_v J_k(v) = - \inf_{\hat{f} \in L^2(\Omega)} (F_1^*(L^* \hat{f}) + F_2^*(-\hat{f})). \tag{1.62}$$

This leads to the following dual problem:
Let $\hat{\psi}$ be defined by

$$-\frac{\partial \hat{\psi}}{\partial t} + A^* \hat{\psi} = 0 \text{ in } \Omega \times (0,T), \quad \hat{\psi}(T) = \hat{f}, \ \hat{\psi} = 0 \text{ on } \Sigma. \tag{1.63}$$

Then the dual problem is to find

$$\inf_{\hat{f} \in L^2(\Omega)} \left[\tfrac{1}{2} \iint_{\mathcal{O}\times(0,T)} \hat{\psi}^2 \, \mathrm{d}x \, \mathrm{d}t - (\hat{f}, y_T) + \frac{1}{2k} \|\hat{f}\|^2 \right], \tag{1.64}$$

or, equivalently,

$$\inf_{\hat{f} \in L^2(\Omega)} \left[\tfrac{1}{2} (\Lambda \hat{f}, \hat{f}) - (\hat{f}, y_T) + \frac{1}{2k} \|\hat{f}\|^2 \right], \tag{1.65}$$

which is in turn equivalent to the *linear* problem (1.57). Problem (1.57), (1.65) has the following variational formulation

$$\left\{ \begin{array}{l} f \in L^2(\Omega) \ \forall \hat{f} \in L^2(\Omega), \text{ we have} \\ \displaystyle\int_\Omega (\Lambda f) \hat{f} \, \mathrm{d}x + \frac{1}{k} \int_\Omega f \hat{f} \, \mathrm{d}x = \int_\Omega y_T \hat{f} \, \mathrm{d}x. \end{array} \right. \tag{1.66}$$

Taking $\hat{f} = f$ in (1.66), we obtain

$$\int_\Omega (\Lambda f) f \, \mathrm{d}x + \frac{1}{k} \|f\|^2 = \int_\Omega y_T f \, \mathrm{d}x. \tag{1.67}$$

We now compare problem (1.64), (1.65) to the problem (1.42), (1.45); it follows from Sections 1.4 and 1.5 that the solution f^* of (1.42), (1.45) satisfies the following *variational inequality*

$$\begin{cases} f^* \in L^2(\Omega)\ \forall \hat{f} \in L^2(\Omega), \text{ we have} \\ \int_\Omega (\Lambda f^*)(\hat{f} - f^*)\,\mathrm{d}x + \beta\|\hat{f}\| - \beta\|f^*\| \geq \int_\Omega y_T(\hat{f} - f^*)\,\mathrm{d}x, \end{cases} \tag{1.68}$$

which implies in turn (take $\hat{f} = 0$ and $\hat{f} = 2f^*$ in (1.68)) that

$$\int_\Omega (\Lambda f^*) f^*\,\mathrm{d}x + \beta\|f^*\| = (y_T, f^*). \tag{1.69}$$

Suppose now that $f = f^*$; it follows then from (1.67), (1.69) that

$$\frac{1}{k}\|f\|^2 = \beta\|f\|,$$

i.e., if $f \neq 0$,

$$k = \|f\|/\beta. \tag{1.70}$$

We propose consequently the following rule:

After a few iterations, where k is given *a priori*, we take k variable with n and defined by

$$k_n = \frac{1}{\beta}\|f^n\|. \tag{1.71}$$

Remark 1.21 It follows from Remark 1.18 and from (1.65) that problem (1.64) is equivalent to (1.57), namely

$$(k^{-1}\mathbf{I} + \Lambda)f = y_T. \tag{1.72}$$

On the other hand, it follows from Section 1.5 that the minimization problem on the right-hand side of (1.33) is equivalent to the 'equation' (it is indeed an inclusion).

$$y_T \in \beta\partial j(f) + \Lambda f, \tag{1.73}$$

where $\partial j(\cdot)$ denotes the *subgradient* (see e.g. Ekeland and Temam (1974) for this notion) of the *convex* functional $j(\cdot)$ defined by

$$j(\hat{f}) = \|\hat{f}\|_{L^2(\Omega)}\ \forall \hat{f} \in L^2(\Omega). \tag{1.74}$$

Intuitively, problem (1.72) being *linear* is easier to solve than (1.73) which is nonlinear, nondifferentiable, etc. In fact, we shall see in Section 1.8 that if one has a method for solving problem (1.72), it can be used in a very simple way to solve problem (1.73).

1.7. L^∞ cost functions and bang–bang controls

We consider the same 'model' problem as before, namely

$$\frac{\partial y}{\partial t} + Ay = v\chi_{\mathcal{O}} \text{ in } \Omega \times (0,T) = Q,\ y(0) = 0,\ y = 0 \text{ on } \Sigma. \tag{1.75}$$

Given $T > 0$ and given $y_T \in L^2(\Omega)$, we consider those control vs such that

$$y(T) \in y_T + \beta B, \tag{1.76}$$

where, in (1.76), β is a positive number and B is the unit ball of $L^2(\Omega)$. Next, we consider the following *control problem*

$$\inf \|v\|_{L^\infty(\mathcal{O}\times(0,T))}, \tag{1.77}$$

where v is subjected to (1.75), (1.76).

A few remarks are in order.

Remark 1.22 This remark is purely technical. The space described by $y(T; v)$ is *dense* in $L^2(\Omega)$ when v spans the space of the C^∞ functions with compact support in $\mathcal{O} \times (0, T)$, so that the infimum in (1.77) is *always* a finite number, *no matter how small* $\beta(> 0)$ *is.*

Remark 1.23 The choice of the L^∞ norm in (1.77) is less convenient than the choice of the L^2 norm, but is not an unreasonable choice. It leads to new difficulties, essentially due to the *nondifferentiability* of the L^∞ norm (and of any power of it). We explain below what to do in order to proceed with this type of cost function, which leads to *bang-bang* type results (see below).

Remark 1.24 Of course, one can more generally consider

$$\inf \|v\|_{L^s(\mathcal{O}\times(0,T))}, \tag{1.78}$$

where s is chosen arbitrarily in $[1, +\infty]$, i.e.

$$1 \le s \le +\infty. \tag{1.79}$$

Indeed, if $s \in (1, +\infty)$ it is more convenient to use $v \to s^{-1}\|v\|^s_{L^s(\mathcal{O}\times(0,T))}$ as the cost function, since it has better *differentiability properties* and does not change the solution of problem (1.78).

Let us consider the case $s = 1$; then for *any* v in $L^1(\mathcal{O}\times(0,T))$ the function $y(T; v)$ belongs to $L^2(\Omega)$ if and only if $d \le 2$ (see, e.g., Ladyzenskaya, Solonnikov and Ural'ceva (1968) for this result). Actually, this does not modify the statement of problem (1.78) (with $s = 1$), since if $d > 2$, we can always restrict ourselves to those controls v in $L^1(\mathcal{O} \times (0, T))$, such that $y(T; v) \in L^2(\Omega)$.

Remark 1.25 There is still another variant that we shall not consider in this article, namely to replace in (1.76) the unit ball B of $L^2(\Omega)$ by the unit ball of $L^r(\Omega)$. We refer to Fabre, Puel and Zuazua (1993) for a discussion of this case.

Remark 1.26 For technical reasons (the explanation for which will appear later on) we are going to consider the problem in the following form

$$\inf \tfrac{1}{2}\|v\|^2_{L^\infty(\mathcal{O}\times(0,T))}, \tag{1.80}$$

or

$$\inf \tfrac{1}{2}\|v\|^2_{L^s(\mathcal{O}\times(0,T))}, \tag{1.81}$$

with v subjected to (1.75), (1.76)

Synopsis In the following, we propose an approximation method for problem (1.80), which leads to: (i) numerical methods; and (ii) connections with one result from Fabre *et al.* (1993).

The results in the above reference have been found by a duality approach, which leads – among other things – to some very interesting formulae; we will present these formulae.

Approximation by penalty and regularization I We begin by considering the following problem

$$\inf J^s_k(v) \tag{1.82}$$

where, in (1.82), the cost function $J^s_k(\cdot)$ is defined by

$$J^s_k(v) = \tfrac{1}{2}\|v\|^2_{L^s} + \tfrac{1}{2}k\|y(T;v) - y_T\|^2_{L^2(\Omega)}, \tag{1.83}$$

and where in (1.83), L^s stands for $L^s(\mathcal{O}\times(0,T))$ and $y(\cdot, v)$ is the solution of (1.75). The idea here is to have $k(>0)$ *large* to 'force' (*penalty*) the final condition $y(T;v) = y_T$, and to have s large, as an approximation of $s = +\infty$ (*regularization*). Problem (1.82) has a *unique* solution and we are going to write the corresponding *optimality conditions*. We can easily verify that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\tfrac{1}{2}\|v+\lambda\hat{v}\|^2_{L^s})|_{\lambda=0} = \|v\|^{2-s}_{L^s}\iint_{\mathcal{O}\times(0,T)} v|v|^{s-2}\hat{v}\,\mathrm{d}x\,\mathrm{d}t \;\forall v,\hat{v}\in L^s. \tag{1.84}$$

The *quadratic* part of $J^s_k(\cdot)$ gives no problem and we verify easily that if we denote by $\nabla J^s_k(\cdot)$ the *derivative* of $J^s_k(\cdot)$ we have

$$\nabla J^s_k(v) \in L^{s'} \;\forall v \in L^s, \text{ with } s' = s/(s-1), \tag{1.85}$$

and, from (1.84),

$$\begin{cases} \displaystyle\iint_{\mathcal{O}\times(0,T)} \nabla J^s_k(v)\hat{v}\,\mathrm{d}x\,\mathrm{d}t = \|v\|^{2-s}_{L^s}\iint_{\mathcal{O}\times(0,T)} v|v|^{s-2}\hat{v}\,\mathrm{d}x\,\mathrm{d}t \\ \displaystyle\qquad\qquad - \iint_{\mathcal{O}\times(0,T)} p\hat{v}\,\mathrm{d}x\,\mathrm{d}t \\ \forall v,\hat{v}\in L^s, \end{cases} \tag{1.86}$$

where, in (1.86), p is the solution of the *adjoint state equation*

$$-\frac{\partial p}{\partial t} + A^*p = 0 \text{ in } Q, \quad p(T) = k(y_T - y(T;v)), \; p = 0 \text{ on } \Sigma, \tag{1.87}$$

with, in (1.87), $y(T;v)$ obtained from v by (1.75); above the exponent s' is the *conjugate* of s, since $1/s + 1/s' = 1$.

Let us denote by u *the* solution of problem (1.82); it satisfies $\nabla J_k^s(u) = 0$, which implies (from (1.86)) that

$$\|u\|_{L^s}^{2-s} u|u|^{s-2} = p\chi_{\mathcal{O}}, \tag{1.88}$$

where, in (1.88), we still denote by p the particular solution of the adjoint system (1.87) corresponding to $v = u$. Relation (1.88) is equivalent to

$$u = \|p\|_{L^s}^{2-s'} p|p|^{s'-2}\chi_{\mathcal{O}}. \tag{1.89}$$

We have therefore obtained the following *optimality system* for problem (1.82):

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = \|p\|_{L^s}^{2-s'} p|p|^{s'-2}\chi_{\mathcal{O}} \text{ in } Q,\ y(0) = 0,\ y = 0 \text{ on } \Sigma, \\ -\dfrac{\partial p}{\partial t} + A^*p = 0 \text{ in } Q,\ p(T) = k(y_T - y(T)),\ p = 0 \text{ on } \Sigma. \end{cases} \tag{1.90}$$

The above result holds for any fixed s arbitrarily large and the same observation applies to k.

The optimality system (1.90) *has a unique solution and the optimal control* u *is given by relation* (1.89).

Approximation by penalty and regularization II Suppose now that $s \to +\infty$, i.e. $s' \to 1$ in (1.90), the parameter k being fixed. We make the assumption (actually it is a *conjecture*; see Fabre *et al.* (1993) for a discussion of this issue) that

$$p \neq 0 \text{ a.e. in } \Omega \times (0,T) \tag{1.91}$$

(except if $p \equiv 0$). Then the limit of (1.90) is given by

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = \|p\|_{L^1}\operatorname{sign} p\chi_{\mathcal{O}} \text{ in } Q,\ y(0) = 0,\ y = 0 \text{ on } \Sigma, \\ -\dfrac{\partial p}{\partial t} + A^*p = 0 \text{ in } Q,\ p(T) = k(y_T - y(T)),\ p = 0 \text{ on } \Sigma. \end{cases} \tag{1.92}$$

Remark 1.27 We observe that (1.92) has been obtained by taking the limit in (1.90) as $s \to +\infty$. This convergence result is not difficult to prove if we suppose that (1.91) holds; see Fabre *et al.* (1993), for further details and results.

Remark 1.28 It follows from (1.92) (or (1.89)) that the *optimal control* u is given by

$$u = \|p\|_{L^1}\operatorname{sign} p\chi_{\mathcal{O}}, \tag{1.93}$$

which is a *bang–bang* result.

Remark 1.29 What has been discussed above is simple thanks to the choice of (1.80) as control problem, which leads in turn to the *regularized* and *regularized–penalized* problems (1.81) and (1.82). This approach and the corresponding results are closely related to those in Fabre *et al.* (1993); in fact, these authors start from the *dual formulation* which is discussed below.

Dual formulation I We can use *duality* as in the L^2 case. We obtain therefore the following duality relation

$$\begin{aligned}\inf_{v\in L^s}[\tfrac{1}{2}\|v\|_{L^s}^2 + \tfrac{1}{2}k\|y(T;v) - y_T\|_{L^2(\Omega)}^2]\\ = -\inf_{\hat f}\left[\tfrac{1}{2}\|\hat\psi\|_{L^{s'}}^2 + \frac{1}{2k}\|\hat f\|_{L^2(\Omega)}^2 - (y_T, \hat f)_{L^2(\Omega)}\right],\end{aligned} \tag{1.94}$$

where, in (1.94), $\hat\psi$ is obtained from $\hat f$ via the solution of

$$-\frac{\partial\hat\psi}{\partial t} + A^*\hat\psi = 0 \text{ in } Q, \quad \hat\psi(T) = \hat f,\ \hat\psi = 0 \text{ on } \Sigma. \tag{1.95}$$

As already mentioned in Remark 1.29, Fabre *et al.* (1993), start from the formulation (1.94), (1.95) *directly* with $s' = 1$; this has to be understood in the following manner: one considers as the *primal problem*

$$\inf_{\hat f\in L^2(\Omega)}\left[\tfrac{1}{2}\|\hat\psi\|_{L^1}^2 + \frac{1}{2k}\|\hat f\|_{L^2(\Omega)}^2 - (y_T, \hat f)_{L^2(\Omega)}\right], \tag{1.96}$$

with $\hat\psi$ still defined by (1.95); then the *dual* problem is

$$\inf_{v\in L^\infty}[\tfrac{1}{2}\|v\|_{L^\infty}^2 + \tfrac{1}{2}k\|y(T;v) - y_T\|_{L^2(\Omega)}^2]. \tag{1.97}$$

Dual formulation II What we want to achieve is (1.76), namely

$$y(T;v) \in y_T + \beta B.$$

Using the *penalized* formulation one obtains $y(T;v)$ 'close' to y_T. In order to have $y(T;v)$ satisfying (1.76) one has to choose k in a suitable fashion. This can be done as follows.

Observe first that, from (1.94), the *dual* problem of problem (1.82) is given by

$$\inf_{\hat f\in L^2(\Omega)}\left[\tfrac{1}{2}\|\hat\psi\|_{L^{s'}}^2 + \frac{1}{2k}\|\hat f\|_{L^2(\Omega)}^2 - (y_T, \hat f)_{L^2(\Omega)}\right]. \tag{1.98}$$

Let us denote by f the solution of problem (1.98); it satisfies (with obvious notation) the following *variational equation* in $L^2(\Omega)$:

$$\left\{\begin{array}{l}\hat f \in L^2(\Omega)\ \forall\hat f \in L^2(\Omega) \text{ we have}\\ \|\psi\|_{L^{s'}}^{2-s'}\displaystyle\iint_{\mathcal{O}\times(0,T)}|\psi|^{s'-2}\psi\hat\psi\,dx\,dt + \frac{1}{k}\int_\Omega f\hat f\,dx = \int_\Omega y_T\hat f\,dx,\end{array}\right. \tag{1.99}$$

which implies in turn that

$$\|\psi\|^2_{L^{s'}} + \frac{1}{k}\|f\|^2_{L^2(\Omega)} = (y_T, f)_{L^2(\Omega)}. \tag{1.100}$$

Consider now the control problem

$$\inf_v \tfrac{1}{2}\|v\|^2_{L^s}, \; v \text{ satisfies } (1.75), (1.76). \tag{1.101}$$

Its dual problem is given by

$$\inf_{\hat{f}\in L^2(\Omega)} [\tfrac{1}{2}\|\hat{\psi}\|^2_{L^{s'}} + \beta\|\hat{f}\|_{L^2(\Omega)} - (y_T, \hat{f})_{L^2(\Omega)}]. \tag{1.102}$$

Denote by f^* the solution of problem (1.102); it satisfies the following *variational inequality* in $L^2(\Omega)$

$$\begin{cases} f^* \in L^2(\Omega)\ \forall \hat{f} \in L^2(\Omega) \text{ we have} \\ \|\psi^*\|^{2-s'}_{L^{s'}} \displaystyle\iint_{\mathcal{O}\times(0,T)} |\psi^*|^{s'-2}\psi^*(\hat{\psi} - \psi^*)\,\mathrm{d}x\,\mathrm{d}t \\ \quad +\beta\|\hat{f}\|_{L^2(\Omega)} - \beta\|f^*\|_{L^2(\Omega)} \geq (y_T, \hat{f} - f^*)_{L^2(\Omega)}. \end{cases} \tag{1.103}$$

Taking successively $\hat{f} = 0$ and $\hat{f} = 2f^*$ in (1.103), we obtain

$$\|\psi^*\|^2_{L^{s'}} + \beta\|f^*\|_{L^2(\Omega)} = (y_T, f^*)_{L^2(\Omega)}. \tag{1.104}$$

The positive member β being given, we look for k such that $f = f^*$, which implies in turn that $\psi = \psi^*$ and therefore that the primal problems (1.82) and (1.101) have the same solution $u(= \|\psi\|^{2-s'}_{L^{s'}}\psi|\psi|^{s'-2}\chi_{\mathcal{O}})$. Suppose that $f = f^*$, it follows then from (1.100) and (1.104) that

$$\frac{1}{k}\|f\|^2_{L^2(\Omega)} = \beta\|f\|_{L^2(\Omega)}.$$

If $\|f\|_{L^2(\Omega)} \neq 0$ we thus have

$$k = \|f\|_{L^2(\Omega)}/\beta. \tag{1.105}$$

From (1.105) we have the following approach to solving problem (1.102) using the solution methods for problem (1.98):

Suppose that we have an *iterative* procedure producing f^1, $f^2, \ldots f^n, \ldots$; we shall use a constant parameter k for several iterations and then a variable one defined by

$$k = \|f^n\|_{L^2(\Omega)}/\beta. \tag{1.106}$$

We shall conclude this section with the following remark.

Remark 1.30 A control problem closely related to those discussed above is the one defined by

$$\inf_{v\in C_f} \tfrac{1}{2}\|y(T;v) - y_T\|^2_{L^2(\Omega)}, \tag{1.107}$$

where, in (1.107), $\mathcal{C}_f$ is the *closed convex* subset of $L^\infty(\mathcal{O} \times (0,T))$ defined by

$$\mathcal{C}_f = \{v \mid v \in L^\infty(\mathcal{O} \times (0,T)), |v(x,t)| \leq C \text{ a.e. in } \mathcal{O} \times (0,T)\}. \quad (1.108)$$

In fact, problem (1.107) is fairly easy to solve if we have solution methods for problem (1.82) with $s = 2$; such methods will be discussed in the following Section 1.8, together with applications to the solution of problems such as (1.107).

1.8. Numerical methods

1.8.1. Generalities. Synopsis.

In this section, we shall address the *numerical solution of the approximate controllability* problems discussed in the preceding sections (the notation of which is kept); we shall start our discussion with the solution of the following two fundamental control problems:

First control problem This is defined by

$$\inf_{v \in \mathcal{U}_f} \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \, dx \, dt \quad (1.109)$$

with $\mathcal{U}_f$ defined by

$$\mathcal{U}_f = \{v \mid v \in L^2(\mathcal{O} \times (0,T)), \; y(T) \in y_T + \beta B\}, \quad (1.110)$$

where, in (1.110), *the target function* y_T is given in $L^2(\Omega)$, B is the unit ball of $L^2(\Omega)$, β is a positive parameter and where the state function y is the solution of the following *parabolic problem*

$$\frac{\partial y}{\partial t} + Ay = v\chi_{\mathcal{O}} \text{ in } Q = \Omega \times (0,T), \quad (1.111)$$

$$y(0) = y_0 (\in L^2(\Omega)), \quad (1.112)$$

$$y = 0 \text{ on } \Sigma = \Gamma \times (0,T). \quad (1.113)$$

Control problem (1.109) has a unique solution.

Second control problem This is defined by

$$\inf_{v \in L^2(\mathcal{O}\times(0,T))} \left[\frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \, dx \, dt + \frac{k}{2} \|y(T) - y_T\|^2_{L^2(\Omega)} \right], \quad (1.114)$$

where, in (1.114), k is *a positive* parameter and y is still defined by (1.111)–(1.113).

Control problem (1.114) has a unique solution.

The solution of the control problems (1.109) and (1.114) can be achieved by methods acting *directly* on the control v; these methods have the advantage of being easy to generalize (in principle) to control problems for *nonlinear* state equations as shown in future sections. In the particular case of problems (1.109) and (1.114) where the state equation (namely (1.111)–(1.113)) is *linear* and the cost functions *quadratic*, instead of solving (1.109) and (1.114) directly, we can solve equivalent problems obtained by applying *Convex Duality Theory*, as already shown in Sections 1.5 and 1.6. In fact, these dual problems can be viewed as *identification* problems for the *final data* of a *backward* (in time) *adjoint equation*, in the spirit of the *Reverse Hilbert Uniqueness Method* (RHUM) introduced in Lions (1988b); from our point of view, these dual problems are better suited to numerical calculations than the original ones (for a discussion concerning the exact and approximate *boundary* controllability of the *heat equation*, which includes numerical methods, see Glowinski (1992b) and Carthel, Glowinski and Lions (1994)).

It follows from Section 1.5 (respectively Section 1.6) that the dual problem to (1.109) (respectively (1.114)) is defined by the following *variational inequality*

$$\begin{cases} f \in L^2(\Omega)\ \forall \hat{f} \in L^2(\Omega) \text{ we have} \\ (\Lambda f, \hat{f} - f)_{L^2(\Omega)} + \beta\|\hat{f}\|_{L^2(\Omega)} - \beta\|f\|_{L^2(\Omega)} \\ \quad \geq (y_T - Y_0(T), \hat{f} - f)_{L^2(\Omega)} \end{cases} \tag{1.115}$$

(respectively by the following *linear* equation

$$(k^{-1}\mathbf{I} + \Lambda)f = y_T - Y_0(T)), \tag{1.116}$$

where in (1.115), (1.116), operator Λ is the one defined in Section 1.5, and where the function Y_0 is defined by

$$\frac{\partial Y_0}{\partial t} + AY_0 = 0 \text{ in } Q, \tag{1.117}$$

$$Y(0) = y_0, \tag{1.118}$$

$$Y_0 = 0 \text{ on } \Sigma. \tag{1.119}$$

In the following subsections we shall discuss the numerical solution of problem (1.116) by methods combining *conjugate gradient algorithms to finite difference and finite element discretizations.* We shall then apply the resulting methodology to the solution of *nonlinear* problem (1.115).

1.8.2. Conjugate gradient solution of problem (1.116).

From now on we shall denote by $(\cdot\,,\cdot)$ and $\|\cdot\|$ the canonical $L^2(\Omega)$-scalar product and $L^2(\Omega)$-norm, respectively. The various approximations of problem (1.116) can be solved by iterative methods closely related to the algorithm discussed in this section.

Writing (1.116) in *variational form* we obtain

$$\begin{cases} f \in L^2(\Omega), \\ k^{-1}(f,\hat{f}) + (\Lambda f, \hat{f}) = (y_T - Y_0(T), \hat{f}) \ \forall \hat{f} \in L^2(\Omega). \end{cases} \tag{1.120}$$

From the *symmetry*, *continuity* and *positive-definiteness* of the bilinear form $\{f,\hat{f}\} \to (\Lambda f, \hat{f})$, the variational problem (1.120) is a particular case of the following general problem

$$\begin{cases} u \in V, \\ a(u,v) = L(v) \ \forall v \in V, \end{cases} \tag{1.121}$$

where:

(i) V is a real *Hilbert space* for the scalar product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$.
(ii) $a : V \times V \to \mathbb{R}$ is bilinear, continuous, symmetric and V-elliptic (i.e. $\exists\alpha > 0$ such that

$$a(v,v) \geq \alpha\|v\|^2 \ \forall v \in V).$$

(iii) $L : V \to \mathbb{R}$ is *linear and continuous.*

If properties (i) to (iii) hold, then problem (1.121) has a *unique* solution (for this result which goes back to Hilbert, see e.g. Lions (1968), Ekeland and Temam (1974), Glowinski (1984)).

Problem (1.121) can be solved by the following *conjugate gradient algorithm:*

$$u^0 \in V \textit{ is given;} \tag{1.122}$$

solve

$$g^0 \in V, \quad (g^0, v) = a(u^0, v) - L(v) \ \forall v \in V, \tag{1.123}$$

and set

$$w^0 = g^0. \quad \square \tag{1.124}$$

For $n \geq 0$, u^n, g^n, w^n *being known, compute* u^{n+1}, g^{n+1}, w^{n+1} *as follows.*

$$\varrho_n = \|g^n\|^2 / a(w^n, w^n) \tag{1.125}$$

and take

$$u^{n+1} = u^n - \varrho_n w^n. \tag{1.126}$$

Solve

$$g^{n+1} \in V, \quad (g^{n+1}, v) = (g^n, v) - \varrho_n a(w^n, v) \ \forall v \in V, \tag{1.127}$$

and compute

$$\gamma_n = \|g^{n+1}\|^2 / \|g^n\|^2, \tag{1.128}$$

$$w^{n+1} = g^{n+1} + \gamma_n w^n. \tag{1.129}$$

Do $n = n + 1$ *and go to* (1.125).

Concerning the convergence of algorithm (1.122)–(1.129) it can be shown (cf. Daniel (1970)) that

$$\|u^n - u\| \leq c\|u^0 - u\| \left(\frac{\sqrt{\nu_a} - 1}{\sqrt{\nu_a} + 1}\right)^n, \tag{1.130}$$

where u is the solution of (1.121), and where the *condition member* ν_a of $a(\cdot,\cdot)$ is defined by $\nu_a = \|A\|\|A^{-1}\|$, where A is the unique operator in $\mathcal{L}(V,V)$ defined by

$$a(v,w) = (Av,w)\ \forall v,w \in V.$$

Application to the solution of problem (1.116) Before applying algorithm (1.122)–(1.129) to the solution of problem (1.116), let us recall the definition of operator Λ; it follows from Section 1.5, relation (1.41), that operator Λ is defined by

$$\Lambda f = \varphi(T), \tag{1.131}$$

where the function φ is obtained from f as follows.

Solve the *backward* equation

$$-\frac{\partial\psi}{\partial t} + A^*\psi = 0 \text{ in } Q, \quad \psi = 0 \text{ on } \Sigma,\ \psi(T) = f, \tag{1.132}$$

and then the *forward* equation

$$\frac{\partial\varphi}{\partial t} + A\varphi = \psi\chi_{\mathcal{O}} \text{ in } Q, \quad \varphi = 0 \text{ on } \Sigma,\ \varphi(0) = 0. \tag{1.133}$$

Applying now algorithm (1.122)–(1.129) to problem (1.116), we obtain the following iterative method (of *conjugated gradient* type);

$$f^0 \textit{ is given in } L^2(\Omega); \tag{1.134}$$

solve first

$$-\frac{\partial p^0}{\partial t} + A^*p^0 = 0 \textit{ in } Q, \quad p^0 = 0 \textit{ on } \Sigma,\ p^0(T) = f^0, \tag{1.135}$$

and set

$$u^0 = p^0\chi_{\mathcal{O}}. \tag{1.136}$$

Solve now

$$\frac{\partial y^0}{\partial t} + Ay^0 = u^0 \textit{ in } Q, \quad y^0 = 0 \textit{ on } \Sigma,\ y^0(0) = y_0, \tag{1.137}$$

compute

$$g^0 = k^{-1}f^0 + y^0(T) - y_T, \tag{1.138}$$

and set

$$w^0 = g^0. \quad \square \tag{1.139}$$

Then, for $n \geq 0$, *assuming that* f^n, g^n, w^n *are known compute* f^{n+1}, g^{n+1}, w^{n+1} *as follows.*

Solve

$$-\frac{\partial \bar{p}^n}{\partial t} + A^* \bar{p}^n = 0 \text{ in } Q, \quad \bar{p}^n = 0 \text{ on } \Sigma, \; \bar{p}^n(T) = w^n \tag{1.140}$$

and set

$$\bar{u}^n = \bar{p}^n \chi_{\mathcal{O}}. \tag{1.141}$$

Solve

$$\frac{\partial \bar{y}^n}{\partial t} + A\bar{y}^n = \bar{u}^n \text{ in } Q, \quad \bar{y}^n = 0 \text{ on } \Sigma, \; \bar{y}^n(0) = 0 \tag{1.142}$$

and compute

$$\bar{g}^n = k^{-1} w^n + \bar{y}^n(T), \tag{1.143}$$

$$\varrho_n = \|g^n\|^2 / (\bar{g}^n, w^n), \tag{1.144}$$

and then

$$f^{n+1} = f^n - \varrho_n w^n, \tag{1.145}$$

$$g^{n+1} = g^n - \varrho_n \bar{g}^n. \tag{1.146}$$

If $\|g^{n+1}\| / \|g^0\| \leq \varepsilon$, *take* $f = f^{n+1}$ *and solve (1.132) to obtain* $u = \psi \chi_{\mathcal{O}}$, *the solution of problem* (1.114); *if the above stopping test is not satisfied, compute*

$$\gamma_n = \|g^{n+1}\|^2 / \|g^n\|^2, \tag{1.147}$$

and then

$$w^{n+1} = g^{n+1} + \gamma_n w^n. \quad \square \tag{1.148}$$

Do $n = n + 1$ *and go to* (1.140).

Remark 1.31 It is fairly easy to show that

$$\|k^{-1}\mathbf{I} + \Lambda\| = k^{-1} + \|\Lambda\|, \quad \|(k^{-1}\mathbf{I} + \Lambda)^{-1}\| = k,$$

implying that the *condition number* of the bilinear from in the left-hand side of (1.120) is equal to $\|\Lambda\| k + 1$. It follows from this result, and from (1.130), that the number of iterations of algorithm (1.134)–(1.148) necessary to obtain convergence varies like $\sqrt{k} \ln \varepsilon^{-1}$ for *large* values of k.

1.8.3. Time discretization of problem (1.116).

The crucial point here is to approximate properly the operator Λ defined by (1.131)–(1.133) in Section 1.8.2. Assuming that T is *bounded* and that operator A is independent of t, we introduce a *time discretization step*, defined by $\Delta t = T/N$, where N is a *positive integer*. Using an *implicit Euler* time discretization scheme, we approximate (1.132) by

$$\psi^{N+1} = f, \quad f \in L^2(\Omega); \tag*{(1.149)$_1$}$$

then, assuming that ψ^{n+1} *is known, we solve the following Dirichlet problem for* $n = N, N-1, \dots, 1$,

$$-\frac{\psi^{n+1}-\psi^n}{\Delta t} + A^*\psi^n = 0 \text{ in } \Omega, \quad \psi^n = 0 \text{ on } \Gamma, \tag{1.149$_2$}$$

where $\psi^n \sim \psi(n\Delta t)(\psi(n\Delta t) : x \to \psi(x, n\Delta t))$. *Next, using similar notation, we approximate* (1.133) *by*

$$\varphi^0 = 0, \tag{1.150$_1$}$$

then assuming that φ^{n-1} *is known, we solve the following Dirichlet problem for* $n = 1, \dots, N$,

$$\frac{\varphi^n - \varphi^{n-1}}{\Delta t} + A\varphi^n = \psi^n \chi_{\mathcal{O}} \text{ in } \Omega, \quad \varphi^n = 0 \text{ on } \Gamma. \tag{1.150$_2$}$$

Finally, we approximate Λ *by* $\Lambda^{\Delta t}$ *defined by*

$$\Lambda^{\Delta t} f = \varphi^N. \quad \square \tag{1.151}$$

From the *ellipticity properties* of A *and* A^* (see Section 1.1), *the Dirichlet* problems $(1.149)_2$ and $(1.150)_2$ have *a unique* solution; we furthermore have the following

Theorem 1.2 Operator $\Lambda^{\Delta t}$ is symmetric and positive semi-definite from $L^2(\Omega)$ into $L^2(\Omega)$.

Proof. Consider a pair $\{f, \hat{f}\} \in L^2(\Omega) \times L^2(\Omega)$. We have then

$$(\Lambda^{\Delta t} f, \hat{f}) = \int_\Omega \varphi^N \hat{\psi}^{N+1}\, dx. \tag{1.152}$$

We also have, since $\varphi^0 = 0$,

$$\Delta t \sum_{n=1}^{N} \left[\varphi^n \left(\frac{\hat{\psi}^{n+1} - \hat{\psi}^n}{\Delta t} \right) + \hat{\psi}^n \left(\frac{\varphi^n - \varphi^{n-1}}{\Delta t} \right) \right] = \varphi^N \hat{\psi}^{N+1}. \tag{1.153}$$

Integrating (1.153) over Ω and taking $(1.149)_2$ into account we obtain

$$\begin{aligned} (\Lambda^{\Delta t} f, \hat{f}) &= \int_\Omega \varphi^N \hat{\psi}^{N+1}\, dx \\ &= \Delta t \sum_{n=1}^{N} \int_\Omega (\varphi^n A^* \hat{\psi}^n - \hat{\psi}^n A \varphi^n)\, dx + \Delta t \sum_{n=1}^{N} \int_{\mathcal{O}} \psi^n \hat{\psi}^n\, dx \\ &= \Delta t \sum_{n=1}^{N} \int_{\mathcal{O}} \psi^n \hat{\psi}^n\, dx, \end{aligned} \tag{1.154}$$

which completes the proof of the theorem. $\square$

Next, we compute the discrete analogue of Y_0 via

$$Y_0^0 = y_0, \tag{1.155$_1$}$$

and for $n = 1, \ldots, N$, assuming that Y_0^{n-1} is known, solve the following (well-posed) elliptic problem

$$\frac{Y_0^n - Y_0^{n-1}}{\Delta t} + AY_0^n = 0 \text{ in } \Omega, \quad Y_0^n = 0 \text{ on } \Gamma. \tag{1.155$_2$}$$

Finally, we approximate problem (1.116) by

$$\begin{cases} f^{\Delta t} \in L^2(\Omega), \\ (k^{-1}f^{\Delta t} + \Lambda^{\Delta t} f^{\Delta t}, \hat{f}) = (y_T - Y_0^N, \hat{f}) \; \forall \hat{f} \in L^2(\Omega). \end{cases} \tag{1.156}$$

Problem (1.156) can be solved by a time discrete analogue of algorithm (1.134)–(1.148).

Remark 1.32 The *Euler schemes* which have been used to time discretize problem (1.116) are only *first-order accurate*; for some applications this may require very small time steps Δt to obtain an acceptable level of accuracy. A simple way to improve this situation is to use second-order schemes like those described in Section 1.8.5 (variants of these schemes have been successfully used to solve boundary controllability problems for the *heat equation* in Carthel *et al.* (1994)).

1.8.4. Full discretization of problem (1.116).

We suppose from now on – and for simplicity – that Ω and $\mathcal{O}$ are *polygonal* domains of $\mathbb{R}^2$ (for nonpolygonal domains Ω and/or $\mathcal{O}$ we shall approximate them by polygonal domains). We introduce then a first *finite element triangulation* $\mathcal{T}_h$ of Ω (h: largest length of the edges of the triangles of $\mathcal{T}_h$) as in Dean, Glowinski and Li (1989), Glowinski, Li and Lions (1990) and Glowinski (1992a); we suppose that both $\bar{\Omega}$ and $\bar{\mathcal{O}}$ are unions of triangles of $\mathcal{T}_h$. Next, we approximate $H^1(\Omega)$, $L^2(\Omega)$ and $H_0^1(\Omega)$ by the following *finite-dimensional* spaces (with P_1 the space of the polynomials in two variables of degree ≤ 1)

$$V_h = \{v_h \mid v_h \in C^0(\bar{\Omega}), v_h|_T \in P_1 \; \forall T \in \mathcal{T}_h\} \tag{1.157}$$

and

$$V_{0h} = \{v_h \mid v_h \in V_h, v_h = 0 \text{ on } \Gamma\} \quad (= V_h \cap H_0^1(\Omega)), \tag{1.158}$$

respectively. We introduce now a second finite element triangulation $\mathcal{T}_H$ of Ω (we may take $\mathcal{T}_h = \mathcal{T}_H$, but the idea here is to have $\mathcal{T}_H$ *coarser* than $\mathcal{T}_h$) and we associate with $\mathcal{T}_H$ the following two finite-dimensional spaces

$$E_H = \{\hat{f}_H \mid \hat{f}_H \in C^0(\bar{\Omega}), \hat{f}_H|_T \in P_1 \; \forall T \in \mathcal{T}_H\}, \tag{1.159}$$

$$E_{0H} = \{\hat{f}_H \mid \hat{f}_H \in E_H, \; \hat{f}_H = 0 \text{ on } \Gamma\} \quad (= E_H \cap H_0^1(\Omega)). \tag{1.160}$$

Since *closure of* $H_0^1(\Omega)$ *in* $L^2(\Omega) = L^2(\Omega)$ we can use either V_h or V_{0h} (respectively E_H or E_{0H}) to approximate $L^2(\Omega)$.

At this stage, it is convenient to (re)introduce $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, the *bilinear* form associated with the elliptic operator A; it is defined by

$$a(y, z) = \langle Ay, z\rangle \ \forall y, z \in H_0^1(\Omega), \tag{1.161}$$

where, in (1.161), $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Similarly we have

$$a(z, y) = \langle A^* y, z\rangle \ \forall y, z \in H_0^1(\Omega). \tag{1.162}$$

From the properties of operator A (see Section 1.1), the above bilinear form is *continuous* over $H_0^1(\Omega) \times H_0^1(\Omega)$ and $H_0^1(\Omega)$-*elliptic.*

We approximate problem (1.116) by

$$\begin{cases} f_{hH}^{\Delta t} \in E_{0H} \ \forall \hat{f}_H \in E_{0H} \text{ we have} \\ \displaystyle\int_\Omega (k^{-1} f_{hH}^{\Delta t} + \Lambda_{hH}^{\Delta t} f_{hH}^{\Delta t}) \hat{f}_H \, dx = \int_\Omega (y_T - Y_{0h}^N) \hat{f}_H \, dx, \end{cases} \tag{1.163}$$

where, in (1.163), Y_{0h}^N is obtained from the *full discretization* of problem (1.117)–(1.119), namely

$$Y_{0h}^0 = y_{0h} \text{ with } y_{0h}(\in V_h) \quad \text{an approximation of } y_0; \tag*{(1.164)$_1$}$$

for $n = 1, \dots, N$, *assuming that* Y_{0h}^{n-1} *is known, compute* Y_{0h}^n *via the solution of the following (approximate and well-posed) elliptic problem.*

$$\begin{cases} Y_{0h}^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{Y_{0h}^n - Y_{0h}^{n-1}}{\Delta t} v_h \, dx + a(Y_{0h}^n, v_h) = 0 \ \forall v_h \in V_{0h}. \end{cases} \tag*{(1.164)$_2$}$$

The operator $\Lambda_{hH}^{\Delta t}$ is defined by

$$\Lambda_{hH}^{\Delta t} f_H = \varphi_h^N \ \forall f_H \in E_{0H}, \tag{1.165}$$

where in order to compute φ_h^N we solve sequentially the following two discrete parabolic problems:

First problem

$$\psi_h^{N+1} = f_H; \tag*{(1.166)$_1$}$$

then for $n = N, N-1, \dots, 1$, *we compute* ψ_h^n *from* ψ_h^{n+1} *via the solution of the following discrete Dirichlet problem*

$$\int_\Omega \frac{\psi_h^n - \psi_h^{n+1}}{\Delta t} v_h \, dx + a(v_h, \psi_h^n) = 0 \ \forall v_h \in V_{0h}; \ \psi_h^n \in V_{0h}. \tag*{(1.166)$_2$}$$

Second problem

$$\varphi_h^0 = 0; \tag*{(1.167)$_1$}$$

for $n = 1, \ldots, N$, *we compute* φ_h^n *from* φ_h^{n-1} *via the solution of the following discrete Dirichlet problem*

$$\int_\Omega \frac{\varphi_h^n - \varphi_h^{n-1}}{\Delta t} v_h \, dx + a(\varphi_h^n, v_h) = \int_O \psi_h^n v_h \, dx \; \forall v_h \in V_{0h}; \; \varphi_h^n \in V_{0h}. \tag{1.167$_2$}$$

The discrete elliptic problems $(1.166)_2$ and $(1.167)_2$ have a *unique* solution (this follows from the properties of the bilinear form $a(\cdot\,,\cdot)$).

Concerning the properties of $\Lambda_{hH}^{\Delta t}$ we can prove the following *fully discrete* analogue of relation (1.154):

$$\int_\Omega (\Lambda_{hH}^{\Delta t} f_H)\hat{f}_H \, dx = \Delta t \sum_{n=1}^{N} \int_O \psi_h^n \hat{\psi}_h^n \, dx \; \forall f_H, \hat{f}_H \in E_{0H}, \tag{1.168}$$

which shows that operator $\Lambda_{hH}^{\Delta t}$ is *symmetric* and *positive semi-definite*, implying in turn that problem (1.163) has a *unique solution* and can be solved by a *conjugate gradient algorithm* (described in the following Section 1.8.5).

Remark 1.33 We can apply the *trapezoidal rule* to evaluate the various $L^2(\Omega)$-scalar products taking place in (1.163), (1.164), (1.166), (1.167) (see Glowinski *et al.* (1990) and Glowinski (1992a) for more details about the use of *numerical integration* in the context of control problems).

Remark 1.34 Instead of E_{0H} we can take the space E_H to approximate problem (1.116); the corresponding approximate problem is still well posed and can be solved by a conjugate gradient algorithm.

1.8.5. Iterative solution of problem (1.163).

From the properties of $\Lambda_{hH}^{\Delta t}$ shown in the previous section, the bilinear form in (1.163) is *symmetric* and *positive definite* (in fact *uniformly* with respect to h, H and Δt). Thus, problem (1.163) can be solved by a *conjugate gradient algorithm* which is a discrete analogue of algorithm (1.134)–(1.148).

Description of the algorithm For simplicity, we shall drop the subscripts h, H and superscript Δt from $f_{hH}^{\Delta t}$.

Initialization

$$f_0 \textit{ is given in } E_{0H}; \tag{1.169}$$

assuming that p_0^{n+1} *is known, solve the following discrete Dirichlet problem for* $n = N, \ldots, 1$

$$\int_\Omega \frac{p_0^n - p_0^{n+1}}{\Delta t} v \, dx + a(v, p_0^n) = 0 \; \forall v \in V_{0h}; \; p_0^n \in V_{0h}, \tag{1.170$_1$}$$

with

$$p_0^{N+1} = f_0, \tag{1.170$_2$}$$

and set

$$u_0^n = p_0^n \,|_{\mathcal{O}} \,. \tag{1.171}$$

Assuming that y_0^{n-1} *is known, solve for* $n = 1, \dots, N$, *the following (well-posed) discrete problem*

$$\int_\Omega \frac{y_0^n - y_0^{n-1}}{\Delta t} v \,\mathrm{d}x + a(y_0^n, v) = \int_{\mathcal{O}} u_0^n v \,\mathrm{d}x \;\forall v \in V_{0h};\; y_0^n \in V_{0h}, \tag{1.172$_1$}$$

with

$$y_0^0 = y_{0h}. \tag{1.172$_2$}$$

Finally, solve the following variational problem

$$\begin{cases} g_0 \in E_{0H}, \\ \displaystyle\int_\Omega g_0 \hat{f} \,\mathrm{d}x = \int_\Omega (k^{-1} f_0 + y_0^N - y_T) \hat{f} \,\mathrm{d}x \;\forall \hat{f} \in E_{0H}, \end{cases} \tag{1.173}$$

and set

$$w_0 = g_0. \quad \square \tag{1.174}$$

Then for $m \geq 0$, *assuming that* f_m, g_m, w_m *are known, compute* f_{m+1}, g_{m+1}, w_{m+1} *as follows.*

Assuming that $\bar{p}_m^{n+1}$ *is known, solve for* $n = N, \dots, 1$, *the following (well-posed) problem*

$$\int_\Omega \frac{\bar{p}_m^n - \bar{p}_m^{n+1}}{\Delta t} v \,\mathrm{d}x + a(v, \bar{p}_m^n) = 0 \;\forall v \in V_{0h};\; \bar{p}_m^n \in V_{0h}, \tag{1.175$_1$}$$

with

$$\bar{p}_m^{N+1} = w_m, \tag{1.175$_2$}$$

and set

$$\bar{u}_m^n = \bar{p}_m^n \,|_{\mathcal{O}} \,. \tag{1.176}$$

Assuming that $\bar{y}_m^{n-1}$ *is known, solve for* $n = 1, \dots, N$, *the following (well-posed) problem*

$$\int_\Omega \frac{\bar{y}_m^n - \bar{y}_m^{n-1}}{\Delta t} v \,\mathrm{d}x + a(\bar{y}_m^n, v) = \int_{\mathcal{O}} \bar{u}_m^n v \,\mathrm{d}x \;\forall v \in V_{0h};\; \bar{y}_m^n \in V_{0h}, \tag{1.177$_1$}$$

with

$$\bar{y}_m^0 = 0. \tag{1.177$_2$}$$

Next, solve

$$\begin{cases} \bar{g}_m \in E_{0H}, \\ \displaystyle\int_\Omega \bar{g}_m \hat{f} \,\mathrm{d}x = \int_\Omega (k^{-1} w_m + \bar{y}_m^N) \hat{f} \,\mathrm{d}x \;\forall \hat{f} \in E_{0H}, \end{cases} \tag{1.178}$$

and compute

$$\rho_m = \frac{\int_\Omega |g_m|^2 \, dx}{\int_\Omega \bar{g}_m w_m \, dx}, \tag{1.179}$$

and then

$$f_{m+1} = f_m - \rho_m w_m, \tag{1.180}$$

$$g_{m+1} = g_m - \rho_m \bar{g}_m. \tag{1.181}$$

If $\|g_{m+1}\|_{L^2(\Omega)}/\|g_0\|_{L^2(\Omega)} \le \varepsilon$, *take* $f = f_{m+1}$ *and solve (1.166) (with* $f_H = f$*) to obtain* $u^n = \psi^n |_{\mathcal{O}}$, *for* $n = 1, \ldots, N$*; if the above stopping test is not satisfied, compute*

$$\gamma_m = \frac{\|g_{m+1}\|^2_{L^2(\Omega)}}{\|g_m\|^2_{L^2(\Omega)}}, \tag{1.182}$$

and then

$$w_{m+1} = g_{m+1} + \gamma_m w_m. \quad \square \tag{1.183}$$

Do $m = m + 1$ *and go to (1.175).*

Remark 1.35 The computer implementation of algorithm (1.169)–(1.183) requires the solution of the *discrete Dirichlet problems* $(1.170)_1$, $(1.172)_1$ and $(1.175)_1$ and $(1.177)_1$; to solve these (linear) problems we can use either *direct methods* (such as *Cholesky's* if the blinear form $a(.,.)$ is *symmetric*) or *iterative methods* (such as *conjugate gradient, relaxation, multigrid*, etc.). To initialize the iterative methods we shall use the solution of the corresponding problem at the previous time step.

A variant of algorithm (1.169)–(1.183) has been employed in Carthel *et al.* (1994), to solve exact and approximate boundary controllability problems for the heat equation; see also Section 2.5 (*Acta Numerica 1995*).

1.8.6. On the use of second-order accurate time discretization schemes for the solution of problem (1.114).

We now complete Remark 1.32 and closely follow Carthel *et al.* (1994, Section 4.6).

1.8.6.1. Generalities. The numerical methods described in Sections 1.8.3 to 1.8.5 rely on a *first-order accurate* time discretization scheme (namely the *backward Euler* scheme). In order to decrease the computational cost for a *given accuracy* (or increase the accuracy for the same computational cost), it makes sense to use higher order time discretization schemes. A natural choice in that direction seems to be the *Crank–Nicolson* scheme (see, e.g. Raviart and Thomas (1988, Ch. 7)) since it is a *one-step, second-order accurate* time discretization scheme, which is, in addition, no more complicated to implement in practice than the backward Euler scheme. Unfortunately, it is well known that the Crank–Nicolson scheme is not well suited (unless one takes Δt of the order of h^2) to simulate *fast transient phenomena* and/or to carry out numerical integration on *long time intervals* $[0, T]$. From these drawbacks a more natural choice is the *two-step implicit* scheme described next which is *second-order accurate*, has much better properties than Crank–Nicolson concerning fast transients and long time intervals, and which is no more complicated to implement in practice than the backward Euler scheme (for a discussion of multistep schemes applied to the time discretization of parabolic problems, see, e.g. Thomee (1990, Section 6)).

1.8.6.2. A second-order accurate time approximation of problem (1.114). In order to solve the control problem (1.114) via the solution of the functional equation (1.116), the crucial point is – again – to properly approximate the operator Λ and the function Y_0 defined in Section 1.8.1.

Approximation of operator Λ Focusing on time discretization, we approximate Λ by $\Lambda^{\Delta t}$ defined as follows (we use the notation of Section 1.8.3).

Let us consider $f \in L^2(\Omega)$, *then*

$$\Lambda^{\Delta t} f = 2\varphi^{N-1} - \varphi^{N-2}, \tag{1.184}$$

where to obtain $\varphi^{N-2}, \varphi^{N-1}$ *we solve first for* $n = N-1, \dots, 1$ *the following (well-posed) Dirichlet problem*

$$\frac{\frac{3}{2}\psi^n - 2\psi^{n+1} + \frac{1}{2}\psi^{n+2}}{\Delta t} + A^*\psi^n = 0 \text{ in } \Omega, \quad \psi^n = 0 \text{ on } \Gamma, \tag{1.185}$$

with

$$\psi^N = 2f, \quad \psi^{N+1} = 4f, \tag{1.186}$$

then, with $\varphi^0 = 0$,

$$\frac{\varphi^1 - \varphi^0}{\Delta t} + (\tfrac{2}{3}A\varphi^1 + \tfrac{1}{3}A\varphi^0) = \tfrac{2}{3}\psi^1\chi_{\mathcal{O}} \text{ in } \Omega, \quad \varphi^1 = 0 \text{ on } \Gamma, \tag{1.187}$$

and, finally, for $n = 2, \dots, N-1$,

$$\frac{\frac{3}{2}\varphi^n - 2\varphi^{n-1} + \frac{1}{2}\varphi^{n-2}}{\Delta t} + A\varphi^n = \psi^n\chi_{\mathcal{O}} \text{ in } \Omega, \quad \varphi^n = 0 \text{ on } \Gamma. \quad \square \tag{1.188}$$

It can be shown that

$$\int_\Omega (\Lambda^{\Delta t} f)\hat{f}\, dx = \Delta t \sum_{n=1}^{N-1} \int_{\mathcal{O}} \psi^n \hat{\psi}^n\, dx \ \forall f,\ \hat{f} \in L^2(\Omega),$$

i.e. Theorem 1.2 still holds for this new operator $\Lambda^{\Delta t}$ (in fact, $\Lambda^{\Delta t}$ has been defined so that the above relation holds; see also Remark 1.36).

Approximation of Y_0 To compute the discrete analogue of Y_0, we take

$Y_0^0 = y_0$ and we solve the Dirichlet problem

$$\frac{Y_0^1 - Y_0^0}{\Delta t} + (\tfrac{2}{3}AY_0^1 + \tfrac{1}{3}AY_0^0) = 0 \text{ in } \Omega, \quad Y_0^1 = 0 \text{ on } \Gamma, \tag{1.189$_1$}$$

and then for $n = 2, \dots, N-1$,

$$\frac{\frac{3}{2}Y_0^n - 2Y_0^{n-1} + \frac{1}{2}Y_0^{n-2}}{\Delta t} + AY_0^n = 0 \text{ in } \Omega, \quad Y_0^n = 0 \text{ on } \Gamma. \tag{1.189$_2$}$$

Approximation of problem (1.116) We approximate problem (1.116) by

$$\begin{cases} f^{\Delta t} \in L^2(\Omega); \ \forall \hat{f} \in L^2(\Omega) \ \text{ we have} \\ (k^{-1}f^{\Delta t} + \Lambda^{\Delta t} f^{\Delta t}, \hat{f})_{L^2(\Omega)} = (y_T - 2Y_0^{N-1} + Y_0^{N-2}, \hat{f})_{L^2(\Omega)}. \end{cases} \tag{1.190}$$

Problem (1.190) can be solved by a discrete analogue of algorithm (1.134)–(1.148). Also, the finite element discretization discussed in Section 1.8.4 can be applied easily to problem (1.190) and the resulting fully discrete problem can be solved by a variant of the conjugate gradient algorithm (1.169)–(1.183).

Remark 1.36 The definition of $\Lambda^{\Delta t}$ via relations (1.184) to (1.188), may look somewhat artificial; in fact, it can be shown that the control obtained via the solution of (1.190) is the *unique* solution of the following (time discrete) control problem:

$$\min_{\{v^n\}_{n=1}^{N-1} \in (L^2(\mathcal{O}))^{N-1}} J^{\Delta t}(v^1, \dots, v^{N-1}), \tag{1.191}$$

where, in (1.191), we have

$$J^{\Delta t}(v_1, \dots, v^{N-1}) = \frac{\Delta t}{2} \sum_{n=1}^{N-1} \int_{\mathcal{O}} |v^n|^2 \, dx + \frac{k}{2} \|2y^{N-1} - y^{N-2} - y_T\|^2_{L^2(\Omega)} \tag{1.192}$$

and where y^{N-2}, y^{N-1} are obtained from $\{v^n\}_{n=1}^{N-1}$ via the solution of the following discrete parabolic problem:

$$y^0 = y_0, \tag{1.193}$$

$$\frac{y^1 - y^0}{\Delta t} + A\left(\tfrac{2}{3}y^1 + \tfrac{1}{3}y^0\right) = \tfrac{2}{3}v^1\chi_{\mathcal{O}} \text{ in } \Omega, \quad y^1 = 0 \text{ on } \Gamma, \tag{1.194}$$

and for $n = 2, \dots, N-1$,

$$\frac{\frac{3}{2}y^n - 2y^{n-1} + \frac{1}{2}y^{n-2}}{\Delta t} + Ay^n = v^n\chi_{\mathcal{O}} \text{ in } \Omega, \quad y^n = 0 \text{ on } \Gamma. \tag{1.195}$$

In principle, $2y^{N-1} - y^{N-2}$ is an $\mathcal{O}(|\Delta t|^2)$ accurate approximate value of $y(T)$ obtained by *extrapolation.*

Remark 1.37 A variant of the previously mentioned second-order time discretization scheme has been successfully applied in Carthel *et al.* (1994, Section 7) to the solution of exact and approximate boundary controllability for the heat equation; see also Section 2.5 (*Acta Numerica 1995*).

1.8.7. Convergence of the approximate solutions of problems (1.114).

In this section, we shall discuss the *convergence* of the solution of the *fully discrete* problem (1.163) – and of the corresponding approximate solution of problem (1.114) – as $\{\Delta t, h, H\} \to 0$. Problem (1.163) has been defined in Section 1.8.4 (whose notation is kept) by

$$\begin{cases} f_{hH}^{\Delta t} \in E_{0H} \ \forall \hat{f}_H \in E_{0H} \quad \text{we have} \\ \displaystyle\int_\Omega (k^{-1} f_{hH}^{\Delta t} + \Lambda_{hH}^{\Delta t} f_{hH}^{\Delta t}) \hat{f}_H \, \mathrm{d}x = \int_\Omega (y_T - Y_{0h}^N) \hat{f}_H \, \mathrm{d}x. \end{cases} \tag{1.196}$$

Concerning the convergence of $\{f_{hH}^{\Delta t}\}_{\{\Delta t, h, H\}}$ as $\{\Delta t, h, H\} \to 0$, we have the following

Theorem 1.3 We suppose that

$$\lim_{h \to 0} \|y_{0h} - y_0\|_{L^2(\Omega)} = 0, \tag{1.197}$$

and

$$\begin{cases} \text{the angles of } \mathcal{T}_h \text{ are uniformly bounded away} \\ \text{from 0 (i.e. } \exists \theta_0 > 0, \text{ such that } \theta \geq \theta_0 \ \forall \theta \text{ angle of } \mathcal{T}_h \ \forall h). \end{cases} \tag{1.198}$$

Then

$$\lim_{\{\Delta t, h, H\} \to 0} \|f_{hH}^{\Delta t} - f\|_{L^2(\Omega)} = 0, \tag{1.199}$$

$$\lim_{\{\Delta t, h, H\} \to 0} \|\psi_{hH}^{\Delta t} \chi_{\mathcal{O}} - u\|_{L^2(\mathcal{O} \times (0,T))} = 0, \tag{1.200}$$

where, in (1.199), f and $f_{hH}^{\Delta t}$ are the solutions of problems (1.116) and (1.163), (1.196), respectively and where, in (1.200), u is the solution of the control problem (1.114) and $\psi_{hH}^{\Delta t} \chi_{\mathcal{O}}$ the discrete control corresponding to $f_{hH}^{\Delta t}$ via (1.166), with $\psi_h^{N+1} = f_{hH}^{\Delta t}$ in $(1.166)_1$.

Proof. To simplify the presentation, we split the proof into several steps.

(i) *Estimates.* Taking $\hat{f}_H = f_{hH}^{\Delta t}$ in (1.196) we obtain, since operator $\Lambda_{hH}^{\Delta t}$ is *positive semi-definite* (see Section 1.8.4, relation (1.168)), that

$$\|f_{hH}^{\Delta t}\|_{L^2(\Omega)} \leq k \|y_T - Y_{0h}^N\|_{L^2(\Omega)} \ \forall \{\Delta t, h, H\}. \tag{1.201}$$

It follows then from standard results on the *finite element approximation of parabolic problems* (see, e.g. Raviart and Thomas (1988, Ch. 7, Section 7.5) and Fujita and Suzuki (1991, Ch. 2, Section 8)) that Properties (1.197), (1.198) imply that

$$\lim_{\{\Delta t,h\}\to 0} \|Y_{0h}^N - Y_0(T)\|_{L^2(\Omega)} = 0, \tag{1.202}$$

where Y_0 is *the* solution of the parabolic problem (1.117)–(1.119). It follows from (1.201) that the family $\{Y_{0h}^N\}_{\{\Delta t,h\}}$ is *bounded* in $L^2(\Omega)$ which implies, in turn, that the right-hand side of (1.201) and therefore

$$\{\|f_{hH}^{\Delta t}\|_{L^2(\Omega)}\}_{\{\Delta t,h,H\}}$$

are *bounded.* Since the family $\{f_{hH}^{\Delta t}\}_{\{\Delta t,h,H\}}$ is bounded in $L^2(\Omega)$ we can extract a subsequence – still denoted by $\{f_{hH}^{\Delta t}\}_{\{\Delta t,h,H\}}$ – such that

$$\lim_{\{\Delta t,h,H\}\to 0} f_{hH}^{\Delta t} = f^* \text{ weakly in } L^2(\Omega). \tag{1.203}$$

(ii) *Weak convergence.* To show that $f^* = f$, it is convenient to introduce Π_H, the $L^2(\Omega)$-projection operator from $L^2(\Omega)$ into E_{0H}; we have

$$\lim_{H\to 0} \|\Pi_H \hat{f} - \hat{f}\|_{L^2(\Omega)} = 0 \ \forall \hat{f} \in L^2(\Omega). \tag{1.204}$$

In Lemma 1.1 which follows later, we shall prove that

$$\lim_{\{\Delta t,h,H\}\to 0} \|\Lambda_{hH}^{\Delta t}\Pi_H \hat{f} - \Lambda \hat{f}\|_{L^2(\Omega)} = 0 \ \forall \hat{f} \in L^2(\Omega). \tag{1.205}$$

It follows then from (1.196), (1.201)–(1.205) and from the *symmetry* of operators Λ and $\Lambda_{hH}^{\Delta t}$ that, $\forall \hat{f} \in L^2(\Omega)$,

$$\begin{aligned}
&\lim_{\{\Delta t,h,H\}\to 0} \int_\Omega (k^{-1} f_{hH}^{\Delta t} + \Lambda_{hH}^{\Delta t} f_{hH}^{\Delta t})\Pi_h \hat{f}\,dx \\
=\ & \lim_{\{\Delta t,h,H\}\to 0} \left[k^{-1}\int_\Omega f_{hH}^{\Delta t}(\Pi_h \hat{f})\,dx + \int_\Omega (\Lambda_{hH}^{\Delta t})\Pi_H \hat{f} f_{hH}^{\Delta t}\,dx\right] \\
=\ & \int_\Omega (k^{-1} f^* + \Lambda f^*)\hat{f}\,dx \\
=\ & \lim_{\{\Delta t,h,H\}\to 0} \int_\Omega (y_T - Y_{0h}^N)\Pi_H \hat{f}\,dx = \int_\Omega (y_T - Y_0(T))\hat{f}\,dx.
\end{aligned}$$

Thus, we have proved (if (1.205) holds) that f^* is a solution of problem (1.116); since (1.116) has a *unique* solution we have $f^* = f$ and also the fact that the *whole* family $\{f_{hH}^{\Delta t}\}_{\{\Delta t,h,H\}}$ converges to f as $\{\Delta t, h, H\} \to 0$.

(iii) *Strong convergence.* Let us introduce $\bar{f}_{hH}^{\Delta t} = f_{hH}^{\Delta t} - \Pi_H f$; we clearly have

$$\lim_{\{\Delta t,h,H\}\to 0} \bar{f}_{hH}^{\Delta t} = 0 \text{ weakly in } L^2(\Omega). \tag{1.206}$$

We also have, $\forall\{\Delta t, h, H\}$,

$$k^{-1}\|\bar{f}_{hH}^{\Delta t}\|_{L^2(\Omega)}^2 \le \int_\Omega (k^{-1}\bar{f}_{hH}^{\Delta t} + \Lambda_{hH}^{\Delta t}\bar{f}_{hH}^{\Delta t})\bar{f}_{hH}^{\Delta t}\,dx. \tag{1.207}$$

Concerning the right-hand side of (1.207) we have, from (1.196),

$$\begin{aligned}(k^{-1}\bar{f}_{hH}^{\Delta t} &+ \Lambda_{hH}^{\Delta t}\bar{f}_{hH}^{\Delta t}, \bar{f}_{hH}^{\Delta t})_{L^2(\Omega)} \\ &= (k^{-1}f_{hH}^{\Delta t} + \Lambda_{hH}^{\Delta t}f_{hH}^{\Delta t}, \bar{f}_{hH}^{\Delta t})_{L^2(\Omega)} - (k^{-1}\Pi_H f + \Lambda_{hH}^{\Delta t}\Pi_H f, \bar{f}_{hH}^{\Delta t}) \\ &= (y_T - Y_{0h}^N, \bar{f}_{hH}^{\Delta t})_{L^2(\Omega)} - (k^{-1}\Pi_H f + \Lambda_{hH}^{\Delta t}\Pi_H f, \bar{f}_{hH}^{\Delta t}.\end{aligned}$$

Taking the limit in the above relations and in (1.207) as $\{\Delta t, h, H\} \to 0$, we obtain from (1.201)–(1.206) that

$$0 \leq \underline{\lim}_{\{\Delta t,h,H\}\to 0}\|\bar{f}_{hH}^{\Delta t}\|_{L^2(\Omega)} \leq \overline{\lim}_{\{\Delta t,h,H\}\to 0}\bar{f}_{hH}^{\Delta t}\|_{L^2(\Omega)} \leq 0; \tag{1.208}$$

we have thus proved that

$$\lim_{\{\Delta t,h,H\}\to 0}\|\bar{f}_{hH}^{\Delta t}\|_{L^2(\Omega)} = 0,$$

which combined with (1.204) (with $\hat{f} = f$) implies in turn the convergence property (1.199).

(iv) *Convergence of the discrete control.* The solution u of the control problem (1.114) satisfies $u = \psi\chi_{\mathcal{O}}$, where ψ is the solution of the parabolic problem (1.132) when $\psi(T) = f$, f being the solution of problem (1.116). Similarly, we associate the solution $f_{hH}^{\Delta t}$ of problem (1.163), (1.196) with the solution $\{\psi_{hH}^n\}_{n=1}^N$ of problem (1.166) when $\psi_h^{N+1} = f_{hH}^{\Delta t}$ in $(1.166)_1$ or, equivalently, the piecewise constant function $\psi_{hH}^{\Delta t}$ of t, defined by

$$\psi_{hH}^{\Delta t} = \sum_{n=1}^{N}\psi_{hH}^n I_n, \tag{1.209}$$

where I_n is the *characteristic function of* $(0,T)\cap((n-1/2)\Delta t, (n+1/2)\Delta t)$.

Since $\lim_{\{\Delta t,h,H\}\to 0}\|f_{hH}^{\Delta t} - f\|_{L^2(\Omega)} = 0$, it follows from Raviart and Thomas (1988), and from Lemma 1.1, that

$$\lim_{\{\Delta t,h,H\}\to 0}\|\psi_{hH}^{\Delta t} - \psi\|_{L^2(Q)} = 0$$

which implies in turn that

$$\lim_{\{\Delta t,h,H\}\to 0}\|\psi_{hH}^{\Delta t}\chi_{\mathcal{O}} - u\|_{L^2(\mathcal{O}\times(0,T))} = 0,$$

i.e. relation (1.200) holds. □

The proof of Theorem 1.3 will be complete once we have proved the following

Lemma 1.1 Suppose that the angle condition (1.198) holds and consider a family $\{\hat{f}_H\}_H$ of E_{0H} such that

$$\lim_{H\to 0}\|\hat{f}_H - \hat{f}\|_{L^2(\Omega)} = 0. \tag{1.210}$$

If with $\hat{f}_H$ we associate $\Lambda_{hH}^{\Delta t}\hat{f}_H, \{\hat{\psi}_h\}_{n=1}^N, \{\hat{\varphi}_h\}_{n=1}^N$ via (1.165)–(1.167), respectively, we then have

$$\lim_{\{\Delta t,h,H\}\to 0} \|\hat{\psi}_{hH}^{\Delta t} - \hat{\psi}\|_{L^2(Q)} = 0, \tag{1.211}$$

$$\lim_{\{\Delta t,h,H\}\to 0} \|\Lambda_{hH}^{\Delta t}\hat{f}_H - \Lambda\hat{f}\|_{L^2(\Omega)} = 0, \tag{1.212}$$

where, in (1.211) $\hat{\psi}_{hH}^{\Delta t}$ is defined from $\{\hat{\psi}_h\}_{n=1}^N$ by (1.209) and where $\hat{\psi}$ is the solution of

$$-\frac{\partial\hat{\psi}}{\partial t} + A^*\hat{\psi} = 0 \text{ in } Q, \quad \hat{\psi} = 0 \text{ on } \Sigma, \quad \hat{\psi}(T) = \hat{f}. \tag{1.213}$$

Proof. (i) Proof of (1.211). For convenience, extend $\{\hat{\psi}_h\}_{n=1}^N$ to $n = 0$ by solving $(1.166)_2$ for $n = 0$, and still denote by $\hat{\psi}_{hH}^{\Delta t}$ the function $\sum_{n=0}^N \hat{\psi}_h^n I_n$; it follows from Raviart and Thomas (1988), that

$$\lim_{\{\Delta t,h,H\}\to 0} \|\hat{\psi}_{hH}^{\Delta t} - \hat{\psi}\|_{L^\infty(0,T;L^2(\Omega))} = 0 \tag{1.214}$$

if we can show that

$$\lim_{\{\Delta t,h,H\}\to 0} \|\hat{\psi}_h^N - \hat{f}\|_{L^2(\Omega)} = 0. \tag{1.215}$$

To show (1.215), observe first that $\hat{\psi}_h^N$ is the unique solution of the discrete elliptic problem

$$\begin{cases} \hat{\psi}_h^N \in V_{0h}, \\ \int_\Omega \hat{\psi}_h^N v_h \,\mathrm{d}x + \Delta t a(v_h, \hat{\psi}_h^N) = \int_\Omega \hat{f}_H v_h \,\mathrm{d}x \ \forall v_h \in V_{0h}. \end{cases} \tag{1.216}$$

Taking $v_h = \hat{\psi}_h^N$ in (1.216), we obtain, from the $H_0^1(\Omega)$ ellipticity of $a(\cdot,\cdot)$ (see Section 1.1), from the Schwarz inequality in $L^2(\Omega)$, and from (1.210), that

$$\|\hat{\psi}_h^N\|_{L^2(\Omega)} \le C \ \forall\{\Delta t, h, H\}, \tag{1.217}$$

$$(\Delta t)^{1/2}\|\hat{\psi}_h^N\|_{H_0^1(\Omega)} \le C \ \forall\{\Delta t, h, H\}, \tag{1.218}$$

where, in (1.217), (1.218) (and in the following), C denotes various quantities independent of $\Delta t, h, H$.

Since from (1.217)), $\{\hat{\psi}_h^N\}_{\{\Delta t,h,H\}}$ is bounded in $L^2(\Omega)$, we can extract a subsequence – still denoted by $\{\hat{\psi}_h^N\}_{\{\Delta t,h,H\}}$ – such that

$$\lim_{\{\Delta t,h,H\}\to 0} \hat{\psi}_h^N = \hat{f}^* \text{ weakly in } L^2(\Omega). \tag{1.219}$$

Consider, next, $v \in \mathcal{D}(\Omega)$ and denote by $r_h v$ the *linear interpolate* of v on $\mathcal{T}_h$; since the *angle condition* (1.198) holds, it follows from, e.g., Ciarlet (1978;

1991), Raviart and Thomas (1988), Glowinski (1984, Appendix 1), that

$$\lim_{h\to 0} \|r_h v - v\|_{H_0^1(\Omega)} = 0 \ \forall v \in \mathcal{D}(\Omega). \tag{1.220}$$

Take now $v_h = r_h v$ in (1.216); it follows then from (1.218), (1.220), and from the continuity of $a(\,.\,,\,.\,)$ over $H_0^1(\Omega) \times H_0^1(\Omega)$ that

$$\left| \int_\Omega \hat{\psi}_h^N r_h v \,\mathrm{d}x - \int_\Omega \hat{f}_H r_h v \,\mathrm{d}x \right| \le C\|v\|_{H_0^1(\Omega)} |\Delta t|^{1/2} \ \forall v \in \mathcal{D}(\Omega). \tag{1.221}$$

Taking the limit in (1.221), as $\{\Delta t, h, H\} \to 0$, it follows then from (1.210), (1.219), (1.220) that

$$\int_\Omega \hat{f}^* v \,\mathrm{d}x = \int_\Omega \hat{f} v \,\mathrm{d}x \ \forall v \in \mathcal{D}(\Omega). \tag{1.222}$$

Since $\mathcal{D}(\Omega)$ is *dense* in $L^2(\Omega)$, it follows from (1.222) that $\hat{f}^* = \hat{f}$ and also that the whole family $\{\hat{\psi}_h^N\}_{\{\Delta t,h,H\}}$ converges weakly to $\hat{f}$. To prove the strong convergence, observe that

$$\begin{aligned}
\int_\Omega |\hat{\psi}_h^N - \hat{f}|^2 \,\mathrm{d}x &= \int_\Omega |\hat{f}|^2 \,\mathrm{d}x - 2\int_\Omega \hat{\psi}_h^N \hat{f} \,\mathrm{d}x + \int_\Omega |\hat{\psi}_h^N|^2 \,\mathrm{d}x \\
&\le \int_\Omega |\hat{f}|^2 \,\mathrm{d}x - 2\int_\Omega \hat{\psi}_h^N \hat{f} \,\mathrm{d}x + \int_\Omega |\hat{\psi}_h^N|^2 \,\mathrm{d}x \\
&\quad + \Delta t a(\hat{\psi}_h^N, \hat{\psi}_h^N) \\
&= \int_\Omega |\hat{f}|^2 \,\mathrm{d}x - 2\int_\Omega \hat{\psi}_h^N \hat{f} \,\mathrm{d}x + \int_\Omega \hat{f}_H \hat{\psi}_h^N \,\mathrm{d}x. \quad (1.223)
\end{aligned}$$

It follows then from (1.210), (1.223) and from the weak convergence of $\{\hat{\psi}_h^N\}_{\{\Delta t,h,H\}}$ to $\hat{f}$ in $L^2(\Omega)$ that the convergence property (1.215) holds; it implies (1.214) and therefore (1.211).

(ii) *Proof of (1.212).* We associate the solution $\hat{\psi}$ of (1.213) with the solution $\hat{\varphi}$ of

$$\frac{\partial \hat{\varphi}}{\partial t} + A\hat{\varphi} = \hat{\psi}|_{\mathcal{O}} \text{ in } Q, \quad \hat{\varphi} = 0 \text{ on } \Sigma, \quad \hat{\varphi}(0) = 0. \tag{1.224}$$

We then have

$$\Lambda \hat{f} = \hat{\varphi}(T). \tag{1.225}$$

Similarly, we associate $\{\hat{\psi}_h^n\}_{n=0}^N$ with $\{\hat{\varphi}_h^n\}_{n=0}^N$ defined by

$$\hat{\varphi}_h^0 = 0, \tag*{(1.226)$_1$}$$

and, for $n = 1, \dots, N$, by the solution of the following discrete elliptic problems

$$\left\{ \begin{array}{l} \hat{\varphi}_h^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{\hat{\varphi}_h^n - \hat{\varphi}_h^{n-1}}{\Delta t} v_h \,\mathrm{d}x + a(\hat{\varphi}_h^n, v_h) = \int_{\mathcal{O}} \hat{\psi}_h^n v_h \,\mathrm{d}x \ \forall v_h \in V_{0h}. \end{array} \right. \tag*{(1.226)$_2$}$$

We have

$$\Lambda_{hH}^{\Delta t} \hat{f}_H = \hat{\varphi}_h^N. \tag{1.227}$$

In order to prove (1.212) it is quite convenient to associate with $\hat{\psi}$ *the family* $\{\hat{\theta}_h^n\}_{n=0}^N$ defined by

$$\hat{\theta}_h^0 = 0, \tag{1.228$_1$}$$

and, for $n = 1, \dots, N$, by the following discrete elliptic problems

$$\begin{cases} \hat{\theta}_h^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{\hat{\theta}_h^n - \hat{\theta}_h^{n-1}}{\Delta t} v_h \, dx + a(\hat{\theta}_h^n, v_h) = \int_\mathcal{O} \hat{\psi}(n\Delta t) v_h \, dx \ \forall v_h \in V_{0h}. \end{cases} \tag{1.228$_2$}$$

Let us define $\hat{\varphi}_{hH}^{\Delta t}$ and $\hat{\theta}_{hH}^{\Delta t}$ by

$$\hat{\varphi}_{hH}^{\Delta t} = \sum_{n=1}^N \hat{\varphi}_h^n I_n, \tag{1.229}$$

$$\hat{\theta}_{hH}^{\Delta t} = \sum_{n=1}^N \hat{\theta}_h^n I_n, \tag{1.230}$$

respectively. Since

$$\hat{\psi} \in C^0([0,T]; L^2(\Omega)),$$

it follows from Raviart and Thomas (1988), that

$$\lim_{\{\Delta t, h, H\} \to 0} \hat{\theta}_{hH}^{\Delta t} = \hat{\varphi} \text{ strongly in } L^2(0,T; H_0^1(\Omega)), \tag{1.231}$$

$$\lim_{\{\Delta t, h, H\} \to 0} \max_{1 \le n \le N} \|\hat{\theta}_h^n - \hat{\varphi}(n\Delta t)\|_{L^2(\Omega)} = 0. \tag{1.232}$$

Actually, similar convergence results hold for $\hat{\varphi}_{hH}^{\Delta t}$. To show them, denote by $\hat{\hat{\varphi}}_{hH}^{\Delta t}$ the difference $\hat{\varphi}_{hH}^{\Delta t} - \hat{\theta}_{hH}^{\Delta t}$; we clearly have

$$\hat{\hat{\varphi}}_h^0 = 0, \tag{1.233$_1$}$$

and for $n = 1, \dots, N$

$$\begin{cases} \hat{\hat{\varphi}}_h^n \in V_{0h} \\ \displaystyle\int_\Omega \frac{\hat{\hat{\varphi}}_h^n - \hat{\hat{\varphi}}_h^{n-1}}{\Delta t} v_h \, dx + a(\hat{\hat{\varphi}}_h^n, v_h) = \int_\mathcal{O} (\hat{\psi}_h^n - \hat{\psi}(n\Delta t)) v_h \, dx \ \forall v_h \in V_{0h}. \end{cases} \tag{1.233$_2$}$$

Take $v_h = \hat{\hat{\varphi}}_h^n$ in (1.233)$_2$ and remember that $a(v,v) \ge \alpha\|v\|^2 \ \forall v \in H_0^1(\Omega)$, with $\alpha > 0$ (see Section 1.1); we then have from the Schwarz inequality in

$L^2(\Omega)$ and from the relation

$$2\alpha\beta \leq c\alpha^2 + c^{-1}\beta^2 \ \forall \alpha, \beta \in \mathbb{R}\, \forall c > 0,$$

that

$$\tfrac{1}{2\Delta t}(\|\hat{\bar{\varphi}}_h^n\|^2_{L^2(\Omega)} - \|\hat{\bar{\varphi}}_h^{n-1}\|^2_{L^2(\Omega)}) + \alpha\|\hat{\bar{\varphi}}_h^n\|^2_{H_0^1(\Omega)} \leq \|\hat{\psi}_h^n - \hat{\psi}(n\Delta t)\|_{L^2(\Omega)}\|\hat{\bar{\varphi}}_h^n\|_{L^2(\Omega)},$$

which implies, in turn, since the injection from $H_0^1(\Omega)$ into $L^2(\Omega)$ is continuous, that $\forall n \geq 1, \forall c > 0$, we have

$$\tfrac{1}{2\Delta t}(\|\hat{\bar{\varphi}}_h^n\|^2_{L^2(\Omega)} - \|\hat{\bar{\varphi}}_h^{n-1}\|^2_{L^2(\Omega)}) + \gamma\|\hat{\bar{\varphi}}_h^n\|^2_{L^2(\Omega)}$$
$$\leq \tfrac{1}{2}\left(\frac{1}{c}\|\hat{\psi}_h^n - \hat{\psi}(n\Delta t)\|^2_{L^2(\Omega)} + c\|\hat{\bar{\varphi}}_h^n\|^2_{L^2(\Omega)}\right), \tag{1.234}$$

where, in (1.234), γ is a *positive constant.*

Taking $c = 2\gamma$ in (1.234), we obtain

$$\|\hat{\bar{\varphi}}_h^n\|^2_{L^2(\Omega)} - \|\hat{\bar{\varphi}}_h^{n-1}\|^2_{L^2(\Omega)} \leq \frac{\Delta t}{2\gamma}\|\hat{\psi}_h^n - \hat{\psi}(n\Delta t)\|^2_{L^2(\Omega)} \ \forall n = 1, \ldots, N,$$

which implies, by summation from $n = 1$ to $n = N$, that

$$\|\hat{\bar{\varphi}}_h^N\|^2_{L^2(\Omega)} \leq \frac{\Delta t}{2\gamma}\sum_{n=1}^{N}\|\hat{\psi}_h^n - \hat{\psi}(n\Delta t)\|^2_{L^2(\Omega)}. \tag{1.235}$$

It follows then from (1.211) and (1.235) that $\lim_{\{\Delta t,h,H\}\to 0}\|\hat{\varphi}_h^N - \hat{\theta}_h^N\|_{L^2(\Omega)} = 0$, which combined with (1.232) implies that

$$\lim_{\{\Delta t,h,H\}\to 0}\|\hat{\varphi}_h^N - \hat{\varphi}(T)\|_{L^2(\Omega)} = 0. \tag{1.236}$$

Finally, relations (1.225), (1.227) and (1.236) imply the convergence result (1.212).

1.8.8. Solution methods for problem (1.115).

In this section, we discuss the solution of the *variational inequality* (1.115), which is *equivalent* to the control problem (1.109) (via a *duality* argument). We observe that (1.115) can also be written as the following nonlinear (multivalued) equation in $L^2(\Omega)$

$$y_T - Y_0(T) \in \Lambda f + \beta\partial j(f), \tag{1.237}$$

where, in (1.237), $\partial j(f)$ denotes the *subgradient* (see, e.g., Ekeland and Temam (1974) for this concept) at f of the *convex functional* $j(\cdot)$ defined by

$$j(\hat{f}) = \|\hat{f}\|_{L^2(\Omega)} \ \forall \hat{f} \in L^2(\Omega).$$

Equation (1.237) strongly suggests the use of *operator splitting methods* like those discussed in, for example, P.L. Lions and Mercier (1979) and Glowinski and Le Tallec (1989). A simple way to derive such methods is to associate with (1.237) a *time-dependent* equation (for a *pseudo-time* τ) such as

$$\begin{cases} \dfrac{\partial f}{\partial \tau} + \Lambda f + \beta \partial j(f) = y_T - Y_0(T), \\ f(0) = f_0 (\in L^2(\Omega)). \end{cases} \tag{1.238}$$

Next, we use time discretization by operator splitting to integrate (1.238) from $\tau = 0$ to $\tau = +\infty$ in order to capture the steady-state solution of (1.238), namely the solution of (1.237).

A natural choice to integrate (1.238) is the *Peaceman–Rachford* scheme (cf. Peaceman and Rachford (1955)), which for the present problem provides

$$f^0 = f_0; \tag{1.239}$$

then, for $k \geq 0$, compute $f^{k+1/2}$ and f^{k+1}, from f^k, by solving

$$\frac{f^{k+1/2} - f^k}{\Delta\tau/2} + \beta \partial j(f^{k+1/2}) + \Lambda f^k = y_T - Y_0(T), \tag{1.240}$$

and

$$\frac{f^{k+1} - f^{k+1/2}}{\Delta\tau/2} + \beta \partial j(f^{k+1/2}) + \Lambda f^{k+1} = y_T - Y_0(T), \tag{1.241}$$

where $\Delta\tau (> 0)$ is a (pseudo) time discretization step. The *convergence* of $\{f^k\}_{k \geq 0}$ to the solution f of (1.115), (1.237) is a direct consequence of Lions and Mercier (1979), Gabay (1982; 1983) and Glowinski and Le Tallec (1989); the convergence results shown in the above references apply to the present problem since operator Λ (respectively function $j(\cdot)$) is *linear, continuous* and *positive definite* (respectively *convex* and *continuous*) over $L^2(\Omega)$.

A variant of this algorithm is given by the following θ-scheme (where $0 < \theta \leq 1/3$; see, e.g., Glowinski and Le Tallec (1989)):

$$f^0 = f_0; \tag{1.242}$$

then, for $k \geq 0$, compute $f^{k+\theta}, f^{k+1-\theta}, f^{k+1}$ from f^k by solving

$$\frac{f^{k+\theta} - f^k}{\theta \Delta\tau} + \beta \partial j(f^{k+\theta}) + \Lambda f^k = y_T - Y_0(T), \tag{1.243}$$

$$\frac{f^{k+1-\theta} - f^{k+\theta}}{(1-2\theta)\Delta\tau} + \beta \partial j(f^{k+\theta}) + \Lambda f^{k+1-\theta} = y_T - Y_0(T), \tag{1.244}$$

$$\frac{f^{k+1} - f^{k+1-\theta}}{\theta \Delta\tau} + \beta \partial j(f^{k+1}) + \Lambda f^{k+1-\theta} = y_T - Y_0(T). \tag{1.245}$$

In practice, it may pay to use a *variable* $\Delta\tau$. Concerning now the solution of the various subproblems in the above two algorithms, we can make the following observations:

(i) Assuming that we know how to solve problems (1.240) and (1.243), (1.245), the functions f^{k+1} and $f^{k+1-\theta}$ are obtained via the solution of linear problems similar to the one the solution of which has been discussed in Sections 1.8.2 to 1.8.7; in particular, we can use the conjugate gradient algorithm (1.169)–(1.183) to solve finite element approximations of problems (1.239) and (1.244).

(ii) Problems (1.240) and (1.243), (1.245) are fairly easy to solve. Consider, for example, problem (1.240); it is clearly equivalent to the following minimization problem

$$\begin{cases} f^{k+1/2} \in L^2(\Omega), \\ J_k(f^{k+1/2}) \leq J_k(v) \ \forall v \in L^2(\Omega), \end{cases} \tag{1.246}$$

with

$$J_k(v) = \tfrac{1}{2}\int_\Omega |v|^2 \,\mathrm{d}x + \beta\tfrac{1}{2}\Delta\tau\|v\|_{L^2(\Omega)} - \int_\Omega f^k v \,\mathrm{d}x \\ -\frac{\Delta\tau}{2}\int_\Omega (y_T - Y_0(T) - \Lambda f^k)v \,\mathrm{d}x \ \forall v \in L^2(\Omega). \tag{1.247}$$

To solve problem (1.246), we define $f_*^{k+1/2}$ as

$$f_*^{k+1/2} = f^k + \tfrac{1}{2}\Delta\tau(y_T - Y_0(T) - \Lambda f^k) \tag{1.248}$$

and observe that the solution of problem (1.246) is clearly of the form

$$f^{k+1/2} = \lambda^{k+1/2} f_*^{k+1/2} \text{ with } \lambda^{k+1/2} \geq 0. \tag{1.249}$$

To obtain $\lambda^{k+1/2}$ we minimize with respect to λ, the polynomial

$$\|f_*^{k+1/2}\|^2_{L^2(\Omega)}(\tfrac{1}{2}\lambda^2 - \lambda) + \tfrac{1}{2}\beta\Delta\tau\|f_*^{k+1/2}\|_{L^2(\Omega)}\lambda (= J_k(\lambda f_*^{k+1/2})).$$

We obtain then (since $\lambda^{k+1/2} \geq 0$)

$$\begin{cases} \lambda^{k+1/2} = 1 - \tfrac{1}{2}\beta\Delta\tau/\|f_*^{k+1/2}\|_{L^2(\Omega)} \quad \text{if } \|f_*^{k+1/2}\|_{L^2(\Omega)} \geq \beta\Delta\tau/2, \\ \lambda^{kl+1/2} = 0 \quad \text{if } \|f_*^{k+1/2}\|_{L^2(\Omega)} < \beta\Delta\tau/2. \end{cases} \tag{1.250}$$

The same method applies to the solution of problems (1.243) and (1.244).

Remark 1.38 Concerning the calculation of f^{k+1} we shall use equation (1.240) to rewrite (1.241) as

$$\frac{f^{k+1} - 2f^{k+1/2} + f^k}{\Delta\tau/2} + \Lambda f^{k+1} = \Lambda f^k, \tag{1.251}$$

which is better suited for practical computations. A similar observation holds for the calculation of $f^{k+1-\theta}$ in (1.244).

1.8.9. Splitting methods for nonquadratic cost functions and control constrained problems.

1.8.9.1. Generalities. In Section 1.7, we have considered control problems such as (or closely related to)

$$\min_{v\in L^s(\mathcal{O}\times(0,T))}\left[\frac{1}{s}\int_{\mathcal{O}\times(0,T)}|v|^s\,\mathrm{d}x\,\mathrm{d}t+\tfrac{1}{2}k\|y(T)-y_T\|^2_{L^2(\Omega)}\right],\tag{1.252}$$

where, in (1.252), $s\in[1,+\infty), k>0$ and where y is defined by (1.111)–(1.113); the case $s=2$ has been treated in Sections 1.8.2 to 1.8.7. Solving (1.252) for large values of s provide solutions close to those obtained with cost functions containing terms such as $\|v\|_{L^\infty(\mathcal{O}\times(0,T))}$.

Another control problem of interest is defined by

$$\min_{v\in\mathcal{C}_f}\tfrac{1}{2}\|y(T)-y_T\|^2_{L^2(\Omega)},\tag{1.253}$$

with

$$\mathcal{C}_f=\{v\mid v\in L^\infty(\mathcal{O}\times(0,T)),|v(x,t)|\le C\ \text{ a.e. in }\mathcal{O}\times(0,T)\}$$

and y still defined by (1.111)–(1.113).

The *convex* set $\mathcal{C}_f$ is clearly *closed* in $L^2(\mathcal{O}\times(0,T))$; we shall denote by $I_{\mathcal{C}_f}$ its *characteristic function* in $L^2(\mathcal{O}\times(0,T))$.

Problem (1.253) is clearly *equivalent* to

$$\min_{v\in L^2(\mathcal{O}\times(0,T))}[I_{\mathcal{C}_f}(v)+\tfrac{1}{2}\|y(T)-y_T\|^2_{L^2(\Omega)}].\tag{1.254}$$

In the following subsections we shall show that problems (1.252) and (1.253), (1.254) are fairly easy to solve if one has a solver for problem (1.114) (i.e. for problem (1.252) when $s=2$).

1.8.9.2. Solution of problem (1.252). Suppose for the time being that $s>1$ and let us denote by $J(\cdot)$ the *strictly convex* functional defined by

$$J(v)=\frac{1}{s}\iint_{\mathcal{O}\times(0,T)}|v|^s\,\mathrm{d}x\,\mathrm{d}t+\tfrac{1}{2}k\|y(T)-y_T\|^2_{L^2(\Omega)},\tag{1.255}$$

where, in (1.255), y is obtained from v via (1.111)–(1.113). Define next $J_1(\cdot)$ and $J_2(\cdot)$ by

$$J_1(v)=\frac{1}{s}\iint_{\mathcal{O}\times(0,T)}|v|^s\,\mathrm{d}x\,\mathrm{d}t\tag{1.256}$$

and

$$J_2(v)=\tfrac{1}{2}k\|y(T)-y_T\|^2_{L^2(\Omega)},\tag{1.257}$$

where y is obtained from v via (1.111)–(1.113), respectively. Both functions are clearly differentiable in $L^s(\mathcal{O} \times (0,T))$ and we have

$$\langle J_1'(v), w\rangle = \iint_{\mathcal{O}\times(0,T)} |v|^{s-2} vw \,\mathrm{d}x\,\mathrm{d}t \;\forall v,\; w \in L^s(\mathcal{O} \times (0,T)), \tag{1.258}$$

$$\langle J_2'(v), w\rangle = -\iint_{\mathcal{O}\times(0,T)} pw \,\mathrm{d}x\,\mathrm{d}t \;\forall v,\; w \in L^s(\mathcal{O} \times (0,T)), \tag{1.259}$$

where, in (1.258), (1.259), $\langle .,.\rangle$ denotes the *duality pairing* between $L^{s'}(\mathcal{O} \times (0,T))$ and $L^s(\mathcal{O} \times (0,T))(s' = s/(s-1))$ and where p is the solution of the *adjoint state equation*

$$-\frac{\partial p}{\partial t} + A^*p = 0 \text{ in } Q, \quad p = 0 \text{ on } \Sigma,\; p(T) = k(y_T - y(T)). \tag{1.260}$$

If u is *the* solution of the control problem (1.252), it is characterized by $J'(u) = 0$, which here takes the following form:

$$J_1'(u) + J_2'(u) = 0. \tag{1.261}$$

In order to solve (1.252), via (1.261), we follow the approach taken in Section 1.8.8 and we associate with (1.261) the following (pseudo) time-dependent problem in $L^s(\mathcal{O} \times (0,T))$:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} + J_1'(u) + J_2'(u) = 0, \\ u(0) = u_0. \end{cases} \tag{1.262}$$

To obtain the *steady-state solution* of (1.262) (i.e. the solution of (1.252), (1.261)) we integrate (1.262) from 0 to $+\infty$ by *operator splitting*; if one uses the *Peaceman–Rachford scheme* (see Section 1.8.8), we obtain

$$u^0 = u_0, \tag{1.263}$$

and for $n \geq 0$, assuming that u^n is known

$$\frac{u^{n+1/2} - u^n}{\frac{1}{2}\Delta\tau} + J_1'(u^{n+1/2}) + J_2'(u^n) = 0, \tag{1.264}$$

$$\frac{u^{n+1} - u^{n+1/2}}{\frac{1}{2}\Delta\tau} + J_1'(u^{n+1/2}) + J_2'(u^{n+1}) = 0. \tag{1.265}$$

Equation (1.264) can also be written

$$\frac{u^{n+1/2} - u^n}{\frac{1}{2}\Delta\tau} + |u^{n+1/2}|^{s-2} u^{n+1/2} - p^n \chi_{\mathcal{O}} = 0, \tag{1.266}$$

where p^n is obtained from u^n via

$$\frac{\partial y^n}{\partial t} + Ay^n = u^n \chi_{\mathcal{O}} \text{ in } Q, \quad y^n = 0 \text{ on } \Sigma, \ y^n(0) = y_0, \tag{1.267}$$

$$-\frac{\partial p^n}{\partial t} + A^* p^n = 0 \text{ in } Q, \quad p^n = 0 \text{ on } \Sigma, \ p^n(T) = k(y_T - y^n(T)). \tag{1.268}$$

We thus obtain $u^{n+1/2}$ from u^n by solving the *nonlinear problem*

$$u^{n+1/2} + \tfrac{1}{2}\Delta\tau |u^{n+1/2}|^{s-2} u^{n+1/2} = u^n + \tfrac{1}{2}\Delta\tau p^n \chi_{\mathcal{O}}. \tag{1.269}$$

Problem (1.269) can be solved *pointwise* in $\mathcal{O} \times (0,T)$; at almost every point of $\mathcal{O} \times (0,T)$ (in practice at the nodes of a *finite difference* or *finite element grid*) we shall have to solve a *one variable* equation of the form

$$\xi + \tfrac{1}{2}\Delta\tau |\xi|^{s-2} \xi = b, \tag{1.270}$$

which has, $\forall b \in \mathbb{R}$, a *unique* solution.

Problem (1.265) is equivalent to the following *minimization problem*

$$\min_{v \in L^2(\mathcal{O}\times(0,T))} j_n(v), \tag{1.271}$$

where $j_n(.)$ is defined by

$$\begin{aligned} j_n(v) = &\tfrac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \, dx \, dt \\ &- \iint_{\mathcal{O}\times(0,T)} (u^{n+1/2} + \tfrac{1}{2}\Delta\tau |u^{n+1/2}|^{s-2} u^{n+1/2}) v \, dx \, dt \\ &+ \tfrac{1}{4} k \Delta\tau \|y(T) - y_T\|^2_{L^2(\Omega)}, \end{aligned} \tag{1.272}$$

with y obtained from v via (1.111)–(1.113). Problem (1.271), (1.272) is a simple variant of problem (1.114); it can therefore be solved by the numerical methods described in Sections 1.8.2 to 1.8.7.

Remark 2.39 From a formal point of view the above method still applies if $s = 1$. In such a case we shall replace (1.269) by the minimization problem

$$\begin{aligned} \min_{v \in L^2(\mathcal{O}\times(0,T))} \Big[&\tfrac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \, dx \, dt \\ &+ \tfrac{1}{2}\Delta\tau \iint_{\mathcal{O}\times(0,T)} |v| \, dx \, dt - \iint_{\mathcal{O}\times(0,T)} (u^n + \tfrac{1}{2}\Delta\tau p^n) v \, dx \, dt \Big] \end{aligned} \tag{1.273}$$

whose solution $u^{n+1/2}$ is given (in *closed form*) by

$$\left\{ \begin{aligned} &u^{n+1/2}(x,t) = 0 \text{ if } |(u^n + \tfrac{1}{2}\Delta\tau p^n)(x,t)| \le \tfrac{1}{2}\Delta\tau, \{x,t\} \in \mathcal{O} \times (0,T), \\ &u^{n+1/2}(x,t) = (u^n + \tfrac{1}{2}\Delta\tau p^n)(x,t) - \tfrac{1}{2}\Delta\tau \text{ sgn } (u^n + \tfrac{1}{2}\Delta\tau p^n)(x,t) \\ &\quad \text{if } |(u^n + \tfrac{1}{2}\Delta\tau p^n)(x,t)| > \tfrac{1}{2}\Delta\tau, \{x,t\} \in \mathcal{O} \times (0,T). \end{aligned} \right. \tag{1.274}$$

Concerning now the calculation of u^{n+1}, we observe that this function is the solution of

$$\frac{u^{n+1} - 2u^{n+1/2} + u^n}{\frac{1}{2}\Delta\tau} + J_2'(u^{n+1}) = J_2'(u^n),$$

which is equivalent to the minimization problem

$$\min_{v \in L^2(\mathcal{O}\times(0,T))} \left[\frac{1}{2}\iint_{\mathcal{O}\times(0,T)} v^2 \,\mathrm{d}x\,\mathrm{d}t + \tfrac{1}{4}\Delta\tau k\|y(T) - y_T\|^2_{L^2(\Omega)} - \iint_{\mathcal{O}\times(0,T)} (2u^{n+1/2} - u^n - \tfrac{1}{2}\Delta\tau p^n) v \,\mathrm{d}x\,\mathrm{d}t\right], \tag{1.275}$$

where y is a function of v via the solution of (1.111)–(1.113). Problem (1.275) is also a variant of problem (1.114).

Remark 1.40 Equation (1.262) and algorithm (1.263)–(1.265) are largely formal if $1 \leq s < 2$; however they make full sense for the discrete analogues of problem (1.252) obtained by finite difference and finite element approximations close to those discussed in Sections 1.8.2 to 1.8.7.

1.8.9.3. Solution of problem (1.253), (1.254). We follow the approach taken in Section 1.8.9.2; we introduce therefore J_1 and J_2 defined by

$$J_1(v) = I_{\mathcal{C}_f}(v), \tag{1.276}$$

and

$$J_2(v) = \tfrac{1}{2}\|y(T) - y_T\|^2_{L^2(\Omega)}, \tag{1.277}$$

y obtained from v via (1.111)–(1.113), respectively. The solution u of problem (1.253), (1.254) is *characterized* therefore by

$$0 \in \partial J_1(u) + J_2'(u) \tag{1.278}$$

where, in (1.278), $\partial J_1(.)$ is the subgradient of $J_1(.)$, and where $J_2'(.)$ is defined by (1.259), (1.260) with $k = 1$.

We associate with (1.278) the following (pseudo) time-dependent problem in $L^2(\mathcal{O} \times (0,T))$:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} + \partial J_1(u) + J_2'(u) = 0, \\ u(0) = u_0 (\in \mathcal{C}_f). \end{cases} \tag{1.279}$$

Applying as in Section 1.8.9.2 the *Peaceman–Rachford scheme*, we obtain

$$u^0 = u_0, \tag{1.280}$$

and for $n \geq 0$, assuming that u^n is known

$$\frac{u^{n+1/2} - u^n}{\frac{1}{2}\Delta\tau} + \partial J_1(u^{n+1/2}) + J_2'(u^n) = 0, \tag{1.281}$$

$$\frac{u^{n+1} - u^{n+1/2}}{\frac{1}{2}\Delta\tau} + \partial J_1(u^{n+1/2}) + J_2'(u^{n+1}) = 0. \tag{1.282}$$

Equation (1.281) is equivalent to the minimization problem

$$\min_{v \in \mathcal{C}_f} \left[\frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \,\mathrm{d}x\,\mathrm{d}t - \iint_{\mathcal{O}\times(0,T)} p^n v \,\mathrm{d}x\,\mathrm{d}t \right], \tag{1.283}$$

where, in (1.283), p^n is obtained from u^n via (1.267), (1.268) with $k = 1$; we have then

$$u^{n+1/2}(x,t) = \min(C, \max(-C, p^n(x,t))), \quad \text{a.e. on } \mathcal{O} \times (0,T). \tag{1.284}$$

Summing (1.281) and (1.282) implies that

$$\frac{u^{n+1} - 2u^{n+1/2} + u^n}{\frac{1}{2}\Delta\tau} + J_2'(u^{n+1}) = J_2'(u^n),$$

which is equivalent to the minimization problem

$$\min_{v \in L^2(\mathcal{O}\times(0,T))} \left[\frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{4}\Delta\tau \|y(T) - y_T\|^2_{L^2(\Omega)} \right.$$
$$\left. - \iint_{\mathcal{O}\times(0,T)} (2u^{n+1/2} - u^n - \tfrac{1}{2}\Delta\tau p^n) v \,\mathrm{d}x\,\mathrm{d}t \right], \tag{1.285}$$

where y is a function of v via (1.111)–(1.113). Problem (1.285) is a variant of problem (1.114).

1.9. Relaxation of controllability

1.9.1. Generalities.

Let $\mathcal{H}$ be a *Hilbert space* and let C be a *linear operator* such that

$$C \in \mathcal{L}(L^2(\Omega); \mathcal{H}), \tag{1.286}$$

and

$$\text{the range of } C \text{ is dense in } \mathcal{H}. \tag{1.287}$$

We consider again the *state equation*

$$\frac{\partial y}{\partial t} + Ay = v\chi_{\mathcal{O}} \text{ in } Q, \quad y(0) = 0, \quad y = 0 \text{ on } \Sigma, \tag{1.288}$$

and we look now for the solution of

$$\inf \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{1.289}$$

for all vs such that

$$Cy(T;v) \in h_T + \beta B_{\mathcal{H}}, \tag{1.290}$$

where h_T is given in $\mathcal{H}$ and where $B_{\mathcal{H}}$ denotes the *unit ball* of $\mathcal{H}$.

Remark 1.41 If $\mathcal{H} = L^2(\Omega)$, C = identity, and if $h_T = y_T$, then problem (1.289) is exactly the control problem discussed before.

1.9.2. Examples of operators C.

Example 1.2 Let ω be an *open* set in Ω and χ_ω be its *characteristic function.* Then

$$Cy = y\chi_\omega \tag{1.291}$$

corresponds to

$$\mathcal{H} = L^2(\omega). \tag{1.292}$$

Here, we want to reach (or to get close to) a given state *on the subset* ω.

Example 1.3 Let $g_1, \dots, g_N$ be N given elements of $L^2(\Omega)$, *linearly independent.* Then

$$Cy = \{(y, g_i)_{L^2(\Omega)}\}_{i=1}^N, \tag{1.293}$$

corresponds to $\mathcal{H} = \mathbb{R}^N$.

The same considerations as in previous sections apply. Let us write down explicitly the dual formulation of (1.289), (1.290), in the particular cases of Examples 1.2 and 1.3.

1.9.3. Dual formulation in the case of Example 1.2.

Let $\hat{f}$ be given in $L^2(\omega)$. We introduce $\hat{\psi}$ defined by

$$-\frac{\partial\hat{\psi}}{\partial t} + A^*\hat{\psi} = 0 \text{ in } Q, \quad \hat{\psi}(T) = \hat{f}\chi_\omega, \quad \hat{\psi} = 0 \text{ on } \Sigma. \tag{1.294}$$

The *dual problem* is then

$$\inf_{\hat{f}\in L^2(\omega)} \left[\tfrac{1}{2}\iint_{\mathcal{O}\times(0,T)} \hat{\psi}^2\, dx\, dt - (\hat{f}, h_T)_{L^2(\omega)} + \beta\|\hat{f}\|_{L^2(\omega)}\right]. \tag{1.295}$$

If f is the (unique) solution of problem (1.295) the solution u of the corresponding control problem (1.289) is given by $u = \psi\chi_{\mathcal{O}\times(0,T)}$, where ψ is the solution of (1.294) corresponding to $\hat{f} = f$.

1.9.4. Dual formulation in the case of Example 1.3.

Let $\hat{\mathbf{f}} = \{\hat{f}_i\}_{i=1}^N$ be given in $\mathbb{R}^N$. We define $\hat{\psi}$ by

$$-\frac{\partial\hat{\psi}}{\partial t} + A^*\hat{\psi} = 0 \text{ in } Q, \quad \hat{\psi}(T) = \sum_{i=1}^N \hat{f}_i g_i, \quad \hat{\psi} = 0 \text{ on } \Sigma. \tag{1.296}$$

The dual problem is then (analogous to (1.295) but with $L^2(\omega)$ replaced by $\mathbb{R}^N$):

$$\inf_{\hat{\mathbf{f}}\in\mathbb{R}^N} \left[\tfrac{1}{2}\iint_{\mathcal{O}\times(0,T)} \hat{\psi}^2\, dx\, dt - (\hat{f}, h_T)_{\mathbb{R}^N} + \beta\|\hat{f}\|_{\mathbb{R}^N}\right]. \tag{1.297}$$

If f is the (unique) solution of problem (1.297) the solution u of the corresponding control problem (1.289) is given by $u = \psi\chi_{\mathcal{O}\times(0,T)}$, where ψ is the solution of (1.296) corresponding to $\hat{f} = f$.

1.9.5. Further comments.

Remark 1.42 We can also consider *time averages* as shown in Lions (1993).

Concerning now the numerical solution of problem (1.289), it can be achieved by numerical methods directly inspired by those discussed in Section 1.8. In particular, it is quite convenient to introduce an operator $\Lambda \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ which will play for problem (1.289) the role played for problems (1.109) and (1.114) by the operator Λ defined in Section 1.5 (see also Section 1.8.2).

Considering, first, Example 1.2, the dual problem (1.295) can also be written as

$$\begin{cases} f \in L^2(\omega), \\ (\Lambda f, \hat{f} - f)_{L^2(\omega)} + \beta\|\hat{f}\|_{L^2(\omega)} - \beta\|f\|_{L^2(\omega)} \\ \quad \geq (h_T, \hat{f} - f)_{L^2(\omega)} \ \forall \hat{f} \in L^2(\omega), \end{cases} \tag{1.298}$$

where, in (1.298), operator Λ is defined as follows

$$\Lambda\hat{f} = \hat{\varphi}(T)\chi_\omega \quad \forall \hat{f} \in L^2(\omega), \tag{1.299}$$

with $\hat{\varphi}(T)$ obtained from $\hat{f}$ via (1.294) and

$$\frac{\partial\hat{\varphi}}{\partial t} + A\hat{\varphi} = \hat{\psi}\chi_{\mathcal{O}} \text{ in } Q, \quad \hat{\varphi}(0) = 0, \ \hat{\varphi} = 0 \text{ on } \Sigma. \tag{1.300}$$

Operator $\Lambda \in \mathcal{L}(L^2(\omega), L^2(\omega))$ and is *symmetric* and *positive definite* over $L^2(\omega)$. The numerical methods discussed in Section 1.8.8 can be easily modified in order to accommodate problem (1.298).

Consider, now, Example 1.3; the dual problem (1.297) can be written as

$$\begin{cases} \mathbf{f} \in \mathbb{R}^N, \\ (\mathbf{\Lambda f}, \hat{\mathbf{f}} - \mathbf{f})_{\mathbb{R}^N} + \beta\|\hat{\mathbf{f}}\|_{\mathbb{R}^N} - \beta\|\mathbf{f}\|_{\mathbb{R}^N} \geq (\mathbf{h}_T, \hat{\mathbf{f}} - \mathbf{f})_{\mathbb{R}^N} \ \forall \hat{\mathbf{f}} \in \mathbb{R}^N, \end{cases} \tag{1.301}$$

where, in (1.301), $\mathbf{\Lambda}$ is the $N \times N$ symmetric and *positive definite matrix* defined by

$$\mathbf{\Lambda} = (\lambda_{ij})_{1\leq i,j\leq N}, \quad \lambda_{ij} = \int_\Omega \varphi_i(T) g_j \, \mathrm{d}x, \tag{1.302}$$

with, in (1.302), φ_i defined from g_i by

$$-\frac{\partial\psi_i}{\partial t} + A^*\psi_i = 0 \text{ in } Q, \quad \psi_i(T) = g_i, \ \psi_i = 0 \text{ on } \Sigma, \tag{1.303}$$

$$\frac{\partial\varphi_i}{\partial t} + A\varphi_i = \psi_i\chi_{\mathcal{O}} \text{ in } Q, \quad \varphi_i(0) = 0, \ \varphi_i = 0 \text{ on } \Sigma. \tag{1.304}$$

Remark 1.43 Problem (1.301) clearly has the 'flavour' of a *Galerkin method* (like those discussed in Section 1.8 to solve problems (1.115) and (1.116)).

To conclude this section we shall discuss a *solution method* for problem (1.301); this method is applicable when N *is not too large*, since it relies on the *explicit* construction of matrix $\mathbf{\Lambda}$. Our solution method is based on the fact that, according to, e.g., Glowinski, Lions and Trémolières (1976, Ch. 2) and (1981, Ch. 2 and Appendix 2), problem (1.301) is *equivalent* to the following *nonlinear system*

$$\begin{cases} \mathbf{\Lambda f} + \beta \mathbf{p} = \mathbf{h}_T \\ (\mathbf{p}, \mathbf{f})_{\mathbb{R}^N} = \|\mathbf{f}\|_{\mathbb{R}^N}, \|\mathbf{p}\|_{\mathbb{R}^N} \leq 1, \end{cases} \tag{1.305}$$

which has a *unique* solution since $\beta > 0$. System (1.305) is in turn *equivalent* to

$$\begin{cases} \mathbf{\Lambda f} + \beta \mathbf{p} = \mathbf{h}_T, \\ \mathbf{p} = P_B(\mathbf{p} + \rho \mathbf{f}) \ \forall \rho > 0, \end{cases} \tag{1.306}$$

where, in (1.306), $P_B : \mathbb{R}^N \to \mathbb{R}^N$ is the *orthogonal projector* from $\mathbb{R}^N$ on the *closed unit ball* B of $\mathbb{R}^N$; we clearly have $\forall \hat{\mathbf{f}} \in \mathbb{R}^N$,

$$P_B(\hat{\mathbf{f}}) = \begin{cases} \hat{\mathbf{f}} & \text{if } \hat{\mathbf{f}} \in B, \\ \hat{\mathbf{f}}/\|\hat{\mathbf{f}}\|_{\mathbb{R}^N} & \text{if } \hat{\mathbf{f}} \notin B. \end{cases}$$

Relations (1.306) suggest the following *iterative* method (of *fixed point* type):

$$\mathbf{p}^0 \in B \text{ is given (we can take, for example, } \mathbf{p}^0 = 0); \tag{1.307}$$

then for $n \geq 0$, assuming that $\mathbf{p}^n$ is known, we compute $\mathbf{f}^n$, and then $\mathbf{p}^{n+1}$, by

$$\mathbf{\Lambda f}^n = \mathbf{h}_T - \beta \mathbf{p}^n, \tag{1.308}$$

$$\mathbf{p}^{n+1} = P_B(\mathbf{p}^n + \rho \mathbf{f}^n). \tag{1.309}$$

Concerning the *convergence* of algorithm (1.307)–(1.309) we then have the following

Proposition 1.2 *Suppose that*

$$0 < \rho < 2\mu_1/\beta, \tag{1.310}$$

where μ_1 is the smallest eigenvalue of matrix $\mathbf{\Lambda}$. Then, $\forall \mathbf{p}^0 \in B$, we have

$$\lim_{n \to +\infty} \{\mathbf{f}^n, \mathbf{p}^n\} = \{\mathbf{f}, \mathbf{p}\}, \tag{1.311}$$

where $\{\mathbf{f}, \mathbf{p}\}$ is the solution of (1.305).

Proof. The convergence result (1.311) is a direct consequence of Glowinski *et al.* (1976, Ch. 2) and (1981, Ch. 2 and Appendix 2) (see also Ciarlet

(1989, Ch. 9)); however, we shall prove it here for the sake of completeness. Introduce therefore $\bar{\mathbf{f}}^n = \mathbf{f}^n - \mathbf{f}$ and $\bar{\mathbf{p}} = \mathbf{p}^n - \mathbf{p}$; we have

$$\mathbf{\Lambda}\bar{\mathbf{f}}^n = -\beta\bar{\mathbf{p}}^n. \tag{1.312}$$

We also have since P_B is a *contraction* $\|\bar{\mathbf{p}}^{n+1}\|_{\mathbb{R}^N} \le \|\bar{\mathbf{p}}^n + \rho\bar{\mathbf{f}}^n\|_{\mathbb{R}^N}$, which implies in turn that

$$\begin{aligned}\|\bar{\mathbf{p}}^{n+1}\|^2_{\mathbb{R}^N} &\le \|\bar{\mathbf{p}}^n\|^2_{\mathbb{R}^N} + 2\rho(\bar{\mathbf{p}}^n, \bar{\mathbf{f}}^n)_{\mathbb{R}^N} + \rho^2\|\bar{\mathbf{f}}^n\|^2_{\mathbb{R}^N} \\ &= \|\bar{\mathbf{p}}^n\|^2_{\mathbb{R}^N} - \frac{2\rho}{\beta}(\mathbf{\Lambda}\bar{\mathbf{f}}^n, \bar{\mathbf{f}}^n) + \rho^2\|\bar{\mathbf{f}}^n\|^2_{\mathbb{R}^N}.\end{aligned} \tag{1.313}$$

It follows from (1.313) that

$$\|\bar{\mathbf{p}}^n\|^2_{\mathbb{R}^N} - \|\bar{\mathbf{p}}^{n+1}\|^2_{\mathbb{R}^N} \ge \frac{2\rho}{\beta}(\mathbf{\Lambda}\bar{\mathbf{f}}^n, \bar{\mathbf{f}}^n)_{\mathbb{R}^N} - \rho^2\|\bar{\mathbf{f}}^n\|^2_{\mathbb{R}^N} \ge \rho\left(\frac{2\mu_1}{\beta} - \rho\right)\|\bar{\mathbf{f}}^n\|^2_{\mathbb{R}^N}, \tag{1.314}$$

where $\mu_1(> 0)$ is the *smallest eigenvalue* of matrix $\mathbf{\Lambda}$. Suppose that (1.310) holds, then the sequence $\{\|\bar{\mathbf{p}}^n\|^2_{\mathbb{R}^N}\}_{n\ge 0}$ is *decreasing*; since it has 0 as a lower bound it converges to some (nonnegative) limit, implying that

$$\lim_{n\to+\infty}(\|\bar{\mathbf{p}}^n\|^2_{\mathbb{R}^N} - \|\bar{\mathbf{p}}^{n+1}\|^2_{\mathbb{R}^N}) = 0. \tag{1.315}$$

Combining (1.310), (1.314), (1.315) we obtain that $\lim_{n\to+\infty}\|\bar{\mathbf{f}}^n\|_{\mathbb{R}^N} = 0$; we have thus shown that $\lim_{n\to+\infty}\mathbf{f}^n = \mathbf{f}$. The convergence of $\{\mathbf{p}^n\}_{n\ge 0}$ to $\mathbf{p}$ follows from the convergence of $\{\mathbf{f}^n\}_{n\ge 0}$ and from (1.308) (or (1.312)).

Remark 1.44 If N is not too large, so that matrix $\mathbf{\Lambda}$ can be constructed (via (1.302)–(1.304)) at a reasonable cost, we shall use the *Cholesky factorization method* (see, e.g., Ciarlet (1989, Ch. 4)) to solve the various systems (1.308). If N is very large, we can expect $\mathbf{\Lambda}$ to be *ill conditioned* and expensive to construct and factorize; therefore, instead of using algorithm (1.307)–(1.309) we suggest solving problem (1.301) by simple variants of the methods used in Section 1.8.8 to solve problem (1.115).

1.10. Pointwise control

1.10.1. Generalities.

A rather natural question in the present framework is to consider situations where in (1.1) the open set $\mathcal{O}$ is replaced by a 'small' set, in particular a set of measure 0. One has then to consider 'functions' which are not in $L^2(\Omega)$ (for a given t).

Many situations can be considered. We confine ourselves here with the case where $\mathcal{O}$ *is reduced to a point*:

$$\mathcal{O} = \{b\}, \quad b \in \Omega. \tag{1.316}$$

Then, if $\delta(x - b)$ denotes the *Dirac measure* at b, the state function y is given by

$$\frac{\partial y}{\partial t} + Ay = v(t)\delta(x-b) \text{ in } Q, \quad y(0)=0,\ y=0 \text{ on } \Sigma. \tag{1.317}$$

In (1.317) the control v is now a function of t only. We shall assume that

$$v \in L^2(0,T). \tag{1.318}$$

Problem (1.317) has a *unique solution*, which is defined by *transposition*, as in Lions and Magenes (1968). It follows from this reference that, *if* $d \leq 3$ one has

$$y \in L^2(Q), \quad \frac{\partial y}{\partial t} \in L^2(0,T;H^{-2}(\Omega)), \tag{1.319}$$

so that

$$t \to y(t;v) \text{ is continuous from } [0,T] \text{ into } H^{-1}(\Omega). \tag{1.320}$$

When v spans $L^2(0,T), y(T;v)$ spans a subspace of $H^{-1}(\Omega)$. Let us look for the orthogonal of (the closure of) this subspace. Let f be given in $H_0^1(\Omega)$ such that

$$\langle y(T;v), f\rangle = 0 \tag{1.321}$$

(where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$). Let ψ be the solution of

$$-\frac{\partial \psi}{\partial t} + A^*\psi = 0 \text{ in } Q, \quad \psi(T)=f,\ \psi=0 \text{ on } \Sigma. \tag{1.322}$$

Then

$$\langle y(T;v), f\rangle = \int_0^T \psi(b,t)v(t)\,\mathrm{d}t. \tag{1.323}$$

Remark 1.45 If $d=1$, the 'function' $\{x,t\} \to v(t)\delta(x-b)$ belongs to

$$L^2(0,T;H^{-1}(\Omega));$$

this property implies in turn that

$$y \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega)), \quad \frac{\partial y}{\partial t} \in L^2(0,T;H^{-1}(\Omega)).$$

Remark 1.46 If $f \in H_0^1(\Omega)$ the solution ψ of (1.322) satisfies

$$\psi \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)),$$

so that, if $d \leq 3$, $\psi(b,t)$ makes sense and belongs to $L^2(0,T)$ (since the injection of $H^2(\Omega)$ into $C^0(\bar{\Omega})$ is continuous) and (1.323) is valid.

It follows from (1.321) and (1.323) that f belongs to the orthogonal of $\{y(T;v)\}$ iff

$$\psi(b,t) = 0. \tag{1.324}$$

Therefore

$$y(T;v) \text{ spans a dense subset of } H^{-1}(\Omega) \text{ when } v \text{ spans } L^2(0,T) \text{ iff } b \text{ is such that (1.324) implies } \psi \equiv 0. \quad (1.325)$$

This is a condition on b, as the following section shows.

1.10.2. On the concept of strategic point. Formulation of a control problem.

We assume that

$$A^* = A, \quad A \text{ independent of } t. \tag{1.326}$$

We introduce the *eigenfunctions* and *eigenvalues* of A (we use here the fact that Ω is *bounded*), i.e.

$$Aw_j = \lambda_j w_j, \quad w_j = 0 \text{ on } \Gamma, \; w_j \neq 0. \tag{1.327}$$

Then (assuming in order to simplify the presentation that the spectrum is *simple*)

$$\psi = \sum_{j=1}^{\infty} (f, w_j) w_j \exp(-\lambda_j (T-t)), \tag{1.328}$$

where, in (1.328), $(\cdot\,,\cdot)$ denotes the scalar product of $L^2(\Omega)$.

We shall say that b is *a strategic point* in Ω if

$$w_j(b) \neq 0 \; \forall j. \tag{1.329}$$

Then (1.324) implies

$$(f, w_j) = 0 \; \forall j,$$

i.e. $f = 0$. *In this case (1.325) is true iff b is a strategic point.*

We assume from now on that (1.325) holds true. We are then looking for the solution of the following control problem

$$\inf_{v \in \mathcal{U}_f} \tfrac{1}{2} \int_0^T v^2 \, \mathrm{d}t, \tag{1.330}$$

with

$$\mathcal{U}_f = \{ v \mid v \in L^2(0,T), \; y(T;v) \in y_T + \beta B_{-1} \}, \tag{1.331}$$

where y_T is given in $H^{-1}(\Omega)$, where $\beta > 0$ and where B_{-1} denotes the unit ball of $H^{-1}(\Omega)$.

1.10.3. Duality results.

The *dual problem* is as follows. One looks (with obvious notation) for the solution of

$$\inf_{\hat{f} \in H_0^1(\Omega)} \left[\tfrac{1}{2} \int_0^T |\hat{\psi}(b,t)|^2 \, \mathrm{d}t - \langle y_T, \hat{f} \rangle + \beta \|\hat{f}\|_{H_0^1(\Omega)} \right]; \tag{1.332}$$

where, in (1.332), $\hat{\psi}$ is obtained from $\hat{f}$ via

$$-\frac{\partial\hat{\psi}}{\partial t} + A^*\hat{\psi} = 0 \text{ on } Q, \quad \hat{\psi}(T) = \hat{f},\ \hat{\psi} = 0 \text{ on } \Sigma. \tag{1.333}$$

The minima in (1.330) and (1.332) are opposite.

If f is the solution of (1.332) then the *optimal control* u (i.e. the solution of (1.330)) is given by

$$u(t) = \psi(b,t), \tag{1.334}$$

where ψ is the solution of (1.333) corresponding to $\hat{f} = f$.

1.10.4. Iterative solution of the dual problem.

From a practical point of view it is convenient to introduce

$$\Lambda \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$$

defined by

$$\Lambda\hat{f} = \hat{\varphi}(T), \tag{1.335}$$

where, in (1.335), $\hat{\varphi}$ is obtained from $\hat{f}$ via (1.333) and

$$\frac{\partial\hat{\varphi}}{\partial t} + A\hat{\varphi} = \hat{\psi}(b,t)\delta(x-b) \text{ in } Q, \quad \hat{\varphi}(0) = 0, \quad \hat{\varphi} = 0 \text{ on } \Sigma. \tag{1.336}$$

We can easily show that

$$\langle \Lambda f_1, f_2 \rangle = \int_0^T \psi_1(b,t)\psi_2(b,t)\,dt \ \forall f_1, f_2 \in H_0^1(\Omega), \tag{1.337}$$

which implies that operator Λ is *self-adjoint* and *positive semi-definite*; operator Λ is positive definite if b is *strategic*.

Combining (1.332) and (1.337) we can rewrite (1.332) as follows

$$\inf_{\hat{f} \in H_0^1(\Omega)} \left[\tfrac{1}{2}\langle \Lambda\hat{f}, \hat{f} \rangle + \beta\|\hat{f}\|_{H_0^1(\Omega)} - \langle y_T, \hat{f} \rangle \right]. \tag{1.338}$$

The minimization problem (1.338) is *equivalent* to the following variational inequality

$$\begin{cases} f \in H_0^1(\Omega), \\ \langle \Lambda f, \hat{f} - f \rangle + \beta\|\hat{f}\|_{H_0^1(\Omega)} - \beta\|f\|_{H_0^1(\Omega)} \geq \langle y_T, \hat{f} - f \rangle \ \forall \hat{f} \in H_0^1(\Omega), \end{cases} \tag{1.339}$$

which can also be written as

$$y_T \in \mathbf{\Lambda} f + \beta\partial j(f), \tag{1.340}$$

where $\partial j(\,\cdot\,)$ is the *subgradient* of the convex functional $j : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$j(\hat{f}) = \|\hat{f}\|_{H_0^1(\Omega)} \ \forall \hat{f} \in H_0^1(\Omega).$$

As seen in previous sections (particularly Section 1.8.8), to solve problem (1.340) we can associate the following *initial value problem* in $H_0^1(\Omega)$ (with $\Delta = \nabla^2$ the *Laplace operator*)

$$\begin{cases} \dfrac{\partial}{\partial\tau}(-\Delta f) + \Lambda f + \beta\partial j(f) = y_T, \\ f(0) = f_0 \end{cases} \tag{1.341}$$

with it and integrate (1.341) from $\tau = 0$ to $\tau = +\infty$, to obtain the steady-state solution of (1.341), i.e. the solution of (1.340).

As in Section 1.8.8, the *Peaceman–Rachford scheme* is well suited to the solution of problem (1.341); we then obtain

$$f^0 = f_0, \tag{1.342}$$

then for $m \geq 0, f^m$ *being known we obtain* $f^{m+1/2}$ *and* f^{m+1} *from*

$$\frac{(-\Delta f^{m+1/2}) - (-\Delta f^m)}{\frac{1}{2}\Delta\tau} + \beta\partial j(f^{m+1/2}) + \Lambda f^m = y_T, \tag{1.343}$$

$$\frac{(-\Delta f^{m+1}) - (-\Delta f^{m+1/2})}{\frac{1}{2}\Delta\tau} + \beta\partial j(f^{m+1/2}) + \Lambda f^{m+1} = y_T. \tag{1.344}$$

Problem (1.343) is equivalent to the *minimization* problem

$$\min_{\hat f \in H_0^1(\Omega)} \Big[\tfrac{1}{2}\int_\Omega |\nabla \hat f|^2\,\mathrm{d}x + \beta\frac{\Delta\tau}{2}\Big(\int_\Omega |\nabla \hat f|^2\,\mathrm{d}x\Big)^{1/2} - \int_\Omega \nabla f^m\cdot\nabla\hat f\,\mathrm{d}x - \frac{\Delta\tau}{2}\langle y_T - \Lambda f^m, \hat f\rangle\Big]. \tag{1.345}$$

Problem (1.345) has a unique solution $f^{m+1/2} \in H_0^1(\Omega)$, which is given by

$$f^{m+1/2} = \lambda^{m+1/2} f_*^{m+1/2}, \tag{1.346}$$

where, in (1.346),

(i) $f_*^{m+1/2}$ is *the* solution of the *Dirichlet* problem

$$\begin{cases} f_*^{m+1/2} \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \nabla f_*^{m+1/2}\cdot\nabla\hat f\,\mathrm{d}x = \int_\Omega \nabla f^m\cdot\nabla\hat f\,\mathrm{d}x \\ \qquad + \dfrac{\Delta\tau}{2}\langle y_T - \Lambda f^m, \hat f\rangle \;\forall \hat f \in H_0^1(\Omega) \end{cases} \tag{1.347}$$

(i.e. $f_*^{m+1/2}$ satisfies $-\Delta(f_*^{m+1/2} - f^m) = \frac{1}{2}\Delta\tau(y_T - \Lambda f^m)$ in Ω, $f_*^{m+1/2} = 0$ on Γ).

(ii) $\lambda^{m+1/2} \geq 0$ and is *the* minimizer over $\mathbb{R}_+$ of the *quadratic polynomial*

$$\lambda \to (\tfrac{1}{2}\lambda^2 - \lambda)\|f_*^{m+1/2}\|^2_{H_0^1(\Omega)} + \beta\frac{\Delta\tau}{2}\lambda\|f_*^{m+1/2}\|_{H_0^1(\Omega)};$$

we then have

$$\lambda^{m+1/2} = \begin{cases} 1 - \beta\frac{\Delta\tau}{2}/\|f_*^{m+1/2}\|_{H_0^1(\Omega)}, & \text{if } \|f_*^{m+1/2}\|_{H_0^1(\Omega)} \geq \beta\frac{\Delta\tau}{2}, \\ 0, \quad \text{if } \|f_*^{m+1/2}\|_{H_0^1(\Omega)} \leq \beta\frac{\Delta\tau}{2}. \end{cases} \tag{1.348}$$

Now, to compute f^{m+1} we observe that (1.343), (1.344) imply

$$-\frac{\Delta(f^{m+1} - 2f^{m+1/2} + f^m)}{\frac{1}{2}\Delta\tau} + \Lambda f^{m+1} = \Lambda f^m,$$

i.e.

$$-\frac{2}{\Delta\tau}\Delta f^{m+1} + \Lambda f^{m+1} = \Lambda f^m - \frac{2}{\Delta\tau}\Delta(2f^{m+1/2} - f^m). \tag{1.349}$$

Problem (1.349) is a particular case of the *'generalized' elliptic problem*

$$-r^{-1}\Delta f + \Lambda f = g, \tag{1.350}$$

where $g \in H^{-1}(\Omega)$ and $r > 0$; the solution of problems like (1.350) will be discussed in the following section.

1.10.5. Solution of problem (1.350).

1.10.5.1. Generalities. From the properties of operators $-\Delta$ and Λ (ellipticity and symmetry) problem (1.350) can be solved by a *conjugate gradient algorithm* like the one discussed in Section 1.8.2. We think, however, that it may be instructive to discuss first a class of control problems closely related to problem (1.330) in which the *dual problems* are of the same form as in (1.350).

Let us consider therefore the following class of *approximate pointwise controllability problems*

$$\min_{v \in L^2(0,T)} \left[\frac{1}{2}\int_0^T v^2\,\mathrm{d}t + \frac{k}{2}\|y(T) - y_T\|^2_{-1}\right], \tag{1.351}$$

obtained by *penalization* of the final condition $y(T) = y_T$. In (1.351)

(i) the *penalty* parameter k is *positive*;

(ii) the function y is obtained from v via (1.317);

(iii) the 'function' y_T belongs to $H^{-1}(\Omega)$;

(iv) the $H^{-1}(\Omega)$-norm $\|\cdot\|_{-1}$ is defined, $\forall g \in H^{-1}(\Omega)$, by

$$\begin{cases} \|g\|_{-1} = \|\tilde{g}\|_{H_0^1(\Omega)} \left(= \left(\int_\Omega |\nabla \tilde{g}|^2 \, dx \right)^{1/2} \right) \text{ with } \tilde{g} \text{ the solution} \\ \text{of the Dirichlet problem } -\Delta \tilde{g} = g \text{ in } \Omega, \ \tilde{g} = 0 \text{ on } \Gamma. \end{cases} \tag{1.352}$$

Problem (1.351) has a *unique* solution y which is characterized by the existence of p belonging to $L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ such that the triple $\{u, y, p\}$ satisfies the following *optimality system*:

$$\frac{\partial y}{\partial t} + Ay = u\delta(x - b) \text{ in } Q, \quad y = 0 \text{ on } \Sigma, \ y(0) = 0, \tag{1.353}$$

$$-\frac{\partial p}{\partial t} + A^* p = 0 \text{ in } Q, \quad p = 0 \text{ on } \Sigma, \tag{1.354_1}$$

$$\begin{cases} p(T) \in H_0^1(\Omega), \\ -\Delta p(T) = k(y_T - y(T)) \text{ in } \Omega, \end{cases} \tag{1.354_2}$$

$$u(t) = p(b,t). \tag{1.355}$$

Let us define $f \in H_0^1(\Omega)$ by $f = p(T)$; it follows then from (1.353)–(1.355) that f is the solution of the (dual) problem

$$-k^{-1}\Delta f + \Lambda f = y_T. \tag{1.356}$$

Concerning the solution of problem (1.351) we have two options: we can use either the primal formulation (1.351) or the dual formulation (1.356). Both approaches will be discussed in the following two sections.

1.10.5.2. Direct solution of problem (1.351). Solving the control problem (1.351) *directly* (i.e. in $L^2(0,T)$) is worth considering for the following reasons:

(i) It can be generalized to pointwise control problems with *nonlinear* state equations.
(ii) The space $L^2(0,T)$ is a space of *one variable* functions, even for multidimensional domains Ω (i.e. $\Omega \subset \mathbb{R}^d$ with $d \geq 2$).
(iii) The structure of the space $L^2(0,T)$ is quite simple making the implementation of conjugate gradient algorithms operating in this space fairly easy.

Let us denote by $J(\cdot)$ the functional in (1.351); the solution u of problem (1.351) satisfies $J'(u) = 0$ where $J'(u)$ is the gradient of the functional $J(\cdot)$ at u. Let us consider $v \in L^2(0,T)$; we can identify $J'(v)$ with an element of $L^2(0,T)$ and we have

$$\int_0^T J'(v) w \, dt = \int_0^T (v(t) - p(b,t)) w(t) \, dt \ \forall v, w \in L^2(0,T), \tag{1.357}$$

where, in (1.357), p is obtained from v via (1.317) and the corresponding *adjoint equation*, namely

$$-\frac{\partial p}{\partial t} + A^* p = 0 \text{ in } Q, \quad p = 0 \text{ on } \Sigma, \tag{1.358$_1$}$$

$$\begin{cases} p(T) \in H_0^1(\Omega), \\ -\Delta p(T) = k(y_T - y(T)) \text{ in } \Omega. \end{cases} \tag{1.358$_2$}$$

Writing $J'(u) = 0$ in *variational form*, namely

$$\begin{cases} u \in L^2(0,T), \\ \displaystyle\int_0^T J'(u) v \,\mathrm{d}t = 0 \ \forall v \in L^2(0,T)), \end{cases}$$

and taking into account the fact that operator $v \to J'(v)$ is *affine* with respect to v (with a linear part associated with an $L^2(0,T)$-elliptic operator) we observe that problem (1.351) is a particular case of problem (1.121) (see Section 1.8.2); it can be solved therefore by the *conjugate gradient algorithm* (1.122)–(1.129). In the particular case considered here this algorithm takes the following form:

$$u^0 \in L^2(0,T) \textit{ is given}; \tag{1.359}$$

solve

$$\frac{\partial y^0}{\partial t} + Ay^0 = u^0 \delta(x-b) \textit{ in } Q, \quad y^0 = 0 \textit{ on } \Sigma, \ y^0(0) = 0, \tag{1.360}$$

then

$$-\frac{\partial p^0}{\partial t} + A^* p^0 = 0 \textit{ in } Q, \quad p^0 = 0 \textit{ on } \Sigma, \tag{1.361$_1$}$$

$$\begin{cases} p^0(T) \in H_0^1(\Omega), \\ -\Delta p^0(T) = k(y_T - y^0(T)) \textit{ in } \Omega, \end{cases} \tag{1.361$_2$}$$

$$\begin{cases} g^0 \in L^2(0,T), \\ \displaystyle\int_0^T g^0(t) v(t) \,\mathrm{d}t = \int_0^T (u^0(t) - p^0(b,t)) v(t) \,\mathrm{d}t \ \forall v \in L^2(0,T), \end{cases} \tag{1.362}$$

and set

$$w^0 = g^0. \quad \square \tag{1.363}$$

Assuming that u^n, g^n, w^n are known, we obtain u^{n+1}, g^{n+1}, w^{n+1} as follows.

Solve

$$\frac{\partial \bar{y}^n}{\partial t} + A\bar{y}^n = w^n \delta(x-b) \textit{ in } Q, \quad \bar{y}^n = 0 \textit{ on } \Sigma, \ \bar{y}^n(0) = 0, \tag{1.364}$$

and then

$$-\frac{\partial \bar{p}^n}{\partial t} + A^* \bar{p}^n = \ in\ Q, \quad \bar{p}^n = 0 \ on\ \Sigma, \tag{1.365$_1$}$$

$$\begin{cases} \bar{p}^n(T) \in H_0^1(\Omega), \\ \Delta \bar{p}^n(T) = k\bar{y}^n(T) \text{ in } \Omega, \end{cases} \tag{1.365$_2$}$$

and

$$\begin{cases} \bar{g}^n \in L^2(0,T), \\ \displaystyle\int_0^T \bar{g}^n v \, \mathrm{d}t = \int_0^T (w^n(t) - \bar{p}^n(b,t))v(t) \, \mathrm{d}t \ \forall v \in L^2(0,T). \end{cases} \tag{1.366}$$

Compute then

$$\rho_n = \int_0^T |g^n|^2 \, \mathrm{d}t \Big/ \int_0^T \bar{g}^n w^n \, \mathrm{d}t, \tag{1.367}$$

and update u^n and g^n by

$$u^{n+1} = u^n - \rho_n w^n, \tag{1.368}$$

and

$$g^{n+1} = g^n - \rho_n \bar{g}^n, \tag{1.369}$$

respectively. If $\|g^{n+1}\|_{L^2(0,T)}/\|g^0\|_{L^2(0,T)} \leq \epsilon$, take $u = u^{n+1}$; if not, compute

$$\gamma_n = \|g^{n+1}\|^2_{L^2(0,T)}/\|g^n\|^2_{L^2(0,T)} \tag{1.370}$$

and update w^n by

$$w^{n+1} = g^{n+1} + \gamma_n w^n. \quad \square \tag{1.371}$$

Do $n = n + 1$ and go to (1.364).

A finite element/finite difference implementation of the above algorithm will be briefly discussed in Section 1.10.6.

1.10.5.3. A duality method for the solution of problem (1.351). Suppose that we can solve the *dual* problem (1.356), then from $f (= p(T))$ we can compute p, via $(1.354)_1$ and obtain the control u via (1.355). Problem (1.356) is equivalent to

$$\begin{cases} f \in H_0^1(\Omega), \\ k^{-1} \displaystyle\int_\Omega \nabla f \cdot \nabla \hat{f} \, \mathrm{d}x + \langle \Lambda f, \hat{f} \rangle = \langle y_T, \hat{f} \rangle \ \forall \hat{f} \in H_0^1(\Omega). \end{cases} \tag{1.372}$$

From the *symmetry, positivity* and *continuity* of Λ (see Section 1.10.4) the *bilinear form* on the left-hand side of (1.372) is *continuous, symmetric* and $H_0^1(\Omega)$-*elliptic* (we have indeed

$$k^{-1} \int_\Omega |\nabla \hat{f}|^2 \, \mathrm{d}x + \langle \Lambda \hat{f}, \hat{f} \rangle \geq k^{-1} \|\hat{f}\|^2_{H_0^1(\Omega)} \ \forall \hat{f} \in H_0^1(\Omega));$$

problem (1.372) (and therefore (1.351)) can be solved by a conjugate gradient algorithm operating this time in $H_0^1(\Omega)$. This algorithm – which is closely related to algorithm (1.134)–(1.148) – is given by

$$f^0 \in H_0^1(\Omega) \textit{ is given}; \tag{1.373}$$

solve

$$-\frac{\partial p^0}{\partial t} + A^* p^0 = 0 \textit{ in } Q, \quad p^0 = 0 \textit{ on } \Sigma,\ p^0(T) = f^0, \tag{1.374}$$

then

$$\frac{\partial y^0}{\partial t} + Ay^0 = p^0(b,t)\delta(x-b) \textit{ in } Q, \quad y^0 = 0 \textit{ on } \Sigma,\ y^0 = 0, \tag{1.375}$$

$$\begin{cases} g^0 \in H_0^1(\Omega), \\ \int_\Omega \nabla g^0 \cdot \nabla \hat{f}\, dx = k^{-1} \int_\Omega \nabla f^0 \cdot \nabla \hat{f}\, dx + \langle y^0(T) - y_T, \hat{f}\rangle\ \forall \hat{f} \in H_0^1(\Omega) \end{cases} \tag{1.376}$$

and set

$$w^0 = g^0. \quad \square \tag{1.377}$$

Assuming that f^n, g^n, w^n are known, we obtain u^{n+1}, g^{n+1}, w^{n+1} as follows.

Solve

$$-\frac{\partial \bar{p}^n}{\partial t} + A^* \bar{p}^n = 0 \textit{ in } Q, \quad \bar{p}^n = 0 \textit{ on } \Sigma,\ \bar{p}^n(T) = w^n, \tag{1.378}$$

then

$$\frac{\partial \bar{y}^n}{\partial t} + A\bar{y}^n = \bar{p}^n(b,t)\delta(x-b) \textit{ in } Q, \quad \bar{y}^n = 0 \textit{ on } \Sigma,\ \bar{y}^n(0) = 0 \tag{1.379}$$

and

$$\begin{cases} \bar{g}^n \in H_0^1(\Omega), \\ \int_\Omega \nabla \bar{g}^n \cdot \nabla \hat{f}\, dx = k^{-1} \int_\Omega \nabla w^n \cdot \nabla \hat{f}\, dx + \langle \bar{y}^n(T), \hat{f}\rangle\ \forall \hat{f} \in H_0^1(\Omega). \end{cases} \tag{1.380}$$

Compute then

$$\rho_n = \int_\Omega |\nabla g^n|^2\, dx \Big/ \int_\Omega \nabla \bar{g}^n \cdot \nabla w^n\, dx, \tag{1.381}$$

and update f^n and g^n by

$$f^{n+1} = f^n - \rho_n w^n, \tag{1.382}$$

and

$$g^{n+1} = g^n - \rho_n \bar{g}^n, \tag{1.383}$$

respectively. If $\|g^{n+1}\|_{H_0^1(\Omega)}/\|g^0\|_{H_0^1(\Omega)} \leq \epsilon$, *take* $f = f^{n+1}$; *if not compute*

$$\gamma_n = \int_\Omega |\nabla g^{n+1}|^2 \, dx \Big/ \int_\Omega |\nabla g^n|^2 \, dx \tag{1.384}$$

and update w^n *by*

$$w^{n+1} = g^{n+1} + \gamma_n w^n. \quad \square \tag{1.385}$$

Do $n = n + 1$ *and go to* (1.378).

Remark 10.47 Concerning the speed of convergence of algorithm (1.373)–(1.385) we have (from Section 1.8.2, relation (1.130)) that the number of iterations necessary to achieve convergence verifies

$$n \leq n_0 \sim \ln \frac{1}{\epsilon} \Big/ \ln \left(\frac{\sqrt{\nu_k} + 1}{\sqrt{\nu_k} - 1} \right), \tag{1.386}$$

where

$$\nu_k = \|k^{-1}\mathbf{I} + \tilde{\Lambda}\| \; \|(k^{-1}\mathbf{I} + \tilde{\Lambda})^{-1}\| \quad (\text{with } \tilde{\Lambda} = (-\Delta)^{-1}\Lambda). \tag{1.387}$$

Since

$$\|k^{-1}\mathbf{I} + \tilde{\Lambda}\| = k^{-1} + \|\tilde{\Lambda}\|, \quad \|(k^{-1}\mathbf{I} + \tilde{\Lambda})^{-1}\| = k,$$

it follows from (1.386) and (1.387) that for *large values* of k we have

$$n \leq n_0 \sim \|\tilde{\Lambda}\|^{1/2} k^{1/2} \ln \epsilon^{-1/2}. \tag{1.388}$$

Similarly, we could have shown that the number of iterations of algorithm (1.359)–(1.371) necessary to obtain convergence also varies like $k^{1/2} \ln \epsilon^{-1/2}$ for large values of k.

From a practical point of view we shall implement finite-dimensional variants of the above algorithms; these variants will be discussed in the following section.

1.10.6. Spacetime discretizations of problems (1.330) and (1.351).

1.10.6.1. Generalities. We shall discuss in this section the *numerical solution* of the pointwise control problems addressed in Sections 1.10.2 to 1.10.5. The approximation methods to be discussed are closely related to those which have been employed in Section 1.8, namely they will combine *time discretizations* by *finite difference* methods to *space discretizations* by *finite element* methods. Since the solution to the control problem (1.330) can be reduced to a sequence of problems such as (1.351), we shall focus our discussion on this last problem.

1.10.6.2. Approximations of control problem (1.351). We now employ the *finite element* spaces V_h and V_{0h} defined as in Section 1.8.4 (the notation of which is mostly kept) we approximate control problem (1.351) as follows.

$$\min_{\mathbf{v}\in\mathbb{R}^N} J_h^{\Delta t}(\mathbf{v}), \tag{1.389}$$

where, in (1.389), we have $\Delta t = T/N, \mathbf{v} = \{v^n\}_{n=1}^N$ and

$$J_h^{\Delta t}(\mathbf{v}) = \frac{\Delta t}{2}\sum_{n=1}^{N}|v^n|^2 + \frac{k}{2}\int_\Omega |\boldsymbol{\nabla}\Phi_h^N|^2\,dx, \tag{1.390}$$

with Φ_h^N obtained from $\mathbf{v}$ via the solution of the following discrete parabolic problem:

$$y_h^0 = 0, \tag{1.391}$$

then for $n = 1, \ldots, N$, assuming that y_h^{n-1} is known, we solve

$$\begin{cases} y_h^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{y_h^n - y_h^{n-1}}{\Delta t} z_h\,dx + a(y_h^n, z_h) = v^n z_h(b)\ \forall z_h \in V_{0h}, \end{cases} \tag{1.392}$$

and finally

$$\begin{cases} \Phi_h^N \in V_{0h}, \\ \displaystyle\int_\Omega \boldsymbol{\nabla}\Phi_h^H \cdot \boldsymbol{\nabla} z_h\,dx = \langle y_T - y_h^N, z_h\rangle\ \forall z_h \in V_{0h}. \end{cases} \tag{1.393}$$

Problems (1.392) (for $n = 1, \ldots, N$) and (1.393) are *well-posed discrete Dirichlet problems* (we recall that $a(z_1, z_2) = \langle Az_1, z_2\rangle\ \forall z_1, z_2 \in H_0^1(\Omega)$).

The discrete control problem (1.389) is *well posed*; its unique solution – denoted by $\mathbf{u}_h^{\Delta t} = \{u^n\}_{n=1}^N$ – is *characterized by*

$$\boldsymbol{\nabla} J_h^{\Delta t}(\mathbf{u}_h^{\Delta t}) = \mathbf{0}, \tag{1.394}$$

where, in (1.394), $\boldsymbol{\nabla} J_h^{\Delta t}$ denotes the gradient of $J_h^{\Delta t}$.

Remark 1.48 The *convergence* of $\mathbf{u}_h^{\Delta t}$, and of the corresponding state vector, to their continuous counterparts is a fairly technical issue. It will not be addressed in this article. On the other hand, we shall address the solution of problem (1.389), via the solution of the equivalent *linear* problem (1.394); this will be the task of the following Sections 1.10.6.3 and 1.10.6.4.

Remark 1.49 The approximate control problem (1.389) relies on a time discretization by an *implicit Euler scheme.* Actually, we can improve accuracy by using, as in Section 1.8.6, a *second-order accurate two-step implicit time discretization scheme.* By merging the techniques described in the present section and in Section 1.8.6 we can easily derive a variant of the approximate problem (1.389) relying on the above second-order accurate time discretization scheme.

1.10.6.3. Iterative solution of the discrete control problem (1.389). I: Calculation of $\boldsymbol{\nabla} J_h^{\Delta t}$. In order to solve via (1.394) the discrete control problem (1.389), by a *conjugate gradient* algorithm, we need to know $\boldsymbol{\nabla} J_h^{\Delta t}(\mathbf{v})\ \forall \mathbf{v} \in \mathbb{R}^N$. To compute $\boldsymbol{\nabla} J_h^{\Delta t}(\mathbf{v})$, we observe that

$$\lim_{\substack{\theta \to 0 \\ \theta \neq 0}} \frac{J_h^{\Delta t}(\mathbf{v}+\theta \mathbf{w}) - J_h^{\Delta t}(\mathbf{v})}{\theta} = (\boldsymbol{\nabla} J_h^{\Delta t}(\mathbf{v}), \mathbf{w})_{\Delta t}\ \forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^N, \tag{1.395}$$

where

$$(\mathbf{v}, \mathbf{w})_{\Delta t} = \Delta t \sum_{n=1}^{N} v^n w^n\ \forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^N \quad (\text{and } \|\mathbf{v}\|_{\Delta t} = (\mathbf{v}, \mathbf{v})_{\Delta t}^{1/2}).$$

Combining (1.390)–(1.393) and (1.395) we can prove that

$$(\boldsymbol{\nabla} J_h^{\Delta t}(\mathbf{v}), \mathbf{w})_{\Delta t} = \Delta t \sum_{n=1}^{N} (v^n - p_h^n(b)) w^n\ \forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^N, \tag{1.396}$$

where the family $\{p_h^n\}_{n=1}^N$ is obtained as the solution to the following *adjoint discrete parabolic problem*:

$$p_h^{N+1} = k\Phi_h^N; \tag{1.397}$$

then, for $n = N, \ldots, 1$, *assuming that* p_h^{N+1} *is known, solve (the well-posed discrete elliptic problem)*

$$\begin{cases} p_h^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{p_h^n - p_h^{n+1}}{\Delta t} z_h \,\mathrm{d}x + a(z_h, p_h^n) = 0\ \forall z_h \in V_{0h}. \end{cases} \tag{1.398}$$

Owing to the importance of relation (1.396), we shall give a short proof of it (of the engineer/physicist type) based on a (formal) *perturbation* analysis: Hence, let us consider a perturbation $\delta\mathbf{v}$ of $\mathbf{v}$; we have then, from (1.390),

$$\begin{aligned} \delta J_h^{\Delta t}(\mathbf{v}) &= (\boldsymbol{\nabla} J_h^{\Delta t}(\mathbf{v}), \delta\mathbf{v})_{\Delta t} \\ &= \Delta t \sum_{n=1}^{N} v^n \delta v^n + k \int_\Omega \boldsymbol{\nabla}\Phi_h^N \cdot \boldsymbol{\nabla}\delta\Phi_h^N \,\mathrm{d}x \end{aligned} \tag{1.399}$$

where, in (1.399), $\delta\Phi_h^N$ is obtained from $\delta\mathbf{v}$ via

$$\delta y_h^0 = 0, \tag{1.400}$$

then for $n = 1, \ldots, N$, we have

$$\begin{cases} \delta y_h^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{\delta y_h^n - \delta y_h^{n-1}}{\Delta t} z_h \,\mathrm{d}x + a(\delta y_h^n, z_h) = \delta v^n z_h(b)\ \forall z_h \in V_{0h}, \end{cases} \tag{1.401}$$

and finally

$$\begin{cases} \delta\Phi_h^N \in V_{0h}, \\ \int_\Omega \nabla\delta\Phi_h^N \cdot \nabla z_h \, dx = -\int_\Omega \delta y_h^N z_h \, dx \ \forall z_h \in V_{0h}. \end{cases} \tag{1.402}$$

Taking $z_h = p_h^n$ in (1.401) we obtain, by summation from $n = 1$ to $n = N$,

$$\begin{aligned} \Delta t \sum_{n=1}^{N} p_h^n(b)\delta v^n &= \Delta t \sum_{n=1}^{N} \int_\Omega \frac{\delta y_h^n - \delta y_h^{n-1}}{\Delta t} p_h^n \, dx + \Delta t \sum_{n=1}^{N} a(\delta y_h^n, p_h^n) \\ &= \int_\Omega p_h^{N+1} \delta y_h^N \, dx \\ &+ \Delta t \sum_{n=1}^{N} \left[\int_\Omega \frac{p_h^n - p_h^{n+1}}{\Delta t} \delta y_h^n \, dx + a(\delta y_h^n, p_h^n) \right]. \end{aligned} \tag{1.403}$$

Since $\{p_h^n\}_{n=1}^{N+1}$ satisfies (1.397), (1.398), it follows from (1.403) that

$$\int_\Omega p_h^{N+1} \delta y_h^N \, dx = \Delta t \sum_{n=1}^{N} p_h^n(b)\delta v^n. \tag{1.404}$$

Taking $z_h = \Phi_h^N$ in (1.402), we obtain from (1.397)

$$k \int_\Omega \nabla\Phi_h^N \cdot \nabla\delta\Phi_h^N \, dx = -k \int_\Omega \delta y_h^N \Phi_h^N \, dx = -\int_\Omega p_h^{N+1} \delta y_h^N \, dx,$$

which combined with (1.399) and (1.402) implies

$$(\nabla J_h^{\Delta t}(\mathbf{v}), \delta\mathbf{v})_{\Delta t} = \Delta t \sum_{n=1}^{N} (v^n - p_h^n(b))\delta v^n. \tag{1.405}$$

Since $\delta\mathbf{v}$ is 'arbitrary', relation (1.405) implies (1.396).

1.10.6.4. Iterative solution of the discrete control problem (1.389). II: Conjugate gradient solution of problem (1.389), (1.394). The discrete control problem (1.389) is equivalent to a linear system (namely (1.394)) which is associated with an $N \times N$ *symmetric* and *positive definite* matrix. Such a problem can therefore be solved by a conjugate gradient algorithm which is a particular case of algorithm (1.122)–(1.129) (see Section 1.8.2) and a variant of algorithm (1.169)–(1.183) (see Section 1.8.5). This algorithm takes the following form:

$$\mathbf{u}_0 = \{u_0^n\}_{n=1}^N \textit{ is given in } \mathbb{R}^N; \tag{1.406}$$

take then

$$y_0^0 = 0, \tag{1.407}$$

and, assuming that y_0^{n-1} is known, solve for $n = 1, \ldots, N$,

$$\begin{cases} y_0^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{y_0^n - y_0^{n-1}}{\Delta t} z_h \, dx + a(y_0^n, z_h) = u_0^n z_h(b) \; \forall z_h \in V_{0h}. \end{cases} \tag{1.408}$$

Solve next

$$\begin{cases} \Phi_0^N \in V_{0h}, \\ \displaystyle\int_\Omega \nabla \Phi_0^N \cdot \nabla z_h \, dx = \langle y_T - y_0^N, z_h \rangle \; \forall z_h \in V_{0h}. \end{cases} \tag{1.409}$$

Finally, take

$$p_0^{N+1} = k\Phi_0^N \tag{1.410}$$

and, assuming that p_0^{n+1} is known, solve for $n = N, \ldots, 1$

$$\begin{cases} p_0^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{p_0^n - p_0^{n+1}}{\Delta t} z_h \, dx + a(z_h, p_0^n) = 0 \; \forall z_h \in V_{0h}. \end{cases} \tag{1.411}$$

Set

$$\mathbf{g}_0 = \{u_0^n - p_0^n(b)\}_n^N = 1 \tag{1.412}$$

and

$$\mathbf{w}_0 = \mathbf{g}_0. \tag{1.413}$$

Then for $m \geq 0$, assuming that $\mathbf{u}_m, \mathbf{g}_m$ and $\mathbf{w}_m$ are known compute $\mathbf{u}_{m+1}, \mathbf{g}_{m+1}$ and $\mathbf{w}_{m+1}$ as follows.

Take

$$\bar{y}_m^0 = 0; \tag{1.414}$$

assuming that $\bar{y}_m^{n-1}$ is known, solve for $n = 1, \ldots, N$

$$\begin{cases} \bar{y}_m^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{\bar{y}_m^n - \bar{y}_m^{n-1}}{\Delta t} z_h \, dx + a(\bar{y}_m^n, z_h) = w_m^n z_h(b) \; \forall z_h \in V_{0h}. \end{cases} \tag{1.415}$$

Solve next

$$\begin{cases} \bar{\Phi}_m^N \in V_{0h}, \\ \displaystyle\int_\Omega \nabla \bar{\Phi}_m^N \cdot \nabla z_h \, dx = -\langle \bar{y}_m^N, z_h \rangle \; \forall z_h \in V_{0h}. \end{cases} \tag{1.416}$$

Finally, take

$$\bar{p}_m^{N+1} = k\bar{\Phi}_m^N, \tag{1.417}$$

and, assuming that $\bar{p}_m^{n+1}$ *is known, solve for* $n = N, \ldots, 1$

$$\begin{cases} \bar{p}_m^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{\bar{p}_m^n - \bar{p}_m^{n+1}}{\Delta t} z_h \, dx + a(z_h, \bar{p}_m^n) = 0 \ \forall z_h \in V_{0h}. \end{cases} \tag{1.418}$$

Set

$$\bar{\mathbf{g}}_m = \{w_m^n - \bar{p}_m^n(b)\}_{n=1}^N. \tag{1.419}$$

Compute

$$\rho_m = \frac{\|\mathbf{g}_m\|_{\Delta t}^2}{(\bar{\mathbf{g}}_m, \mathbf{w}_m)_{\Delta t}}, \tag{1.420}$$

and update $\mathbf{u}_m$ *and* $\mathbf{g}_m$ *by*

$$\mathbf{u}_{m+1} = \mathbf{u}_m - \rho_m \mathbf{w}_m, \tag{1.421}$$

$$\mathbf{g}_{m+1} = \mathbf{g}_m - \rho_m \bar{\mathbf{g}}_m, \tag{1.422}$$

respectively. If $\|\mathbf{g}_{m+1}\|_{\Delta t}/\|\mathbf{g}_0\|_{\Delta t} \leq \epsilon$ *take* $\mathbf{u}_h^{\Delta t} = \mathbf{u}^{m+1}$*; else, compute*

$$\gamma_m = \|\mathbf{g}_{m+1}\|_{\Delta t}^2 / \|\mathbf{g}_m\|_{\Delta t}^2, \tag{1.423}$$

and update $\mathbf{w}_m$ *by*

$$\mathbf{w}_{m+1} = \mathbf{g}_{m+1} + \gamma_m \mathbf{w}_m. \quad \square \tag{1.424}$$

Do $m = m + 1$ *and go to* (1.414).

Remark 1.50 Algorithm (1.406)–(1.424) is a discrete analogue of algorithm (1.359)–(1.371).

1.10.6.5. Approximation of the dual problem (1.356). It was shown in Section 1.10.5 that there is *equivalence* between the *primal* control problem (1.351) and its *dual* problem (1.356). We shall discuss now the approximation of problem (1.356). There is no difficulty in adapting problem (1.356) to the (backward Euler scheme based) approximation methods discussed in Sections 1.8.3 to 1.8.5 for the solution of problem (1.116). Therefore, to avoid tedious repetitions we shall focus our discussion on an approximation of problem (1.356) which is based on a time discretization by the two-step backward implicit scheme considered in Section 1.8.6 (whose notation is kept); for simplicity, we shall take $H = h$ and $E_{0h} = V_{0h}$.

We approximate the *dual problem* (1.356), (1.372) by

$$\begin{cases} f_h^{\Delta t} \in V_{0h}, \\ k^{-1} \displaystyle\int_\Omega \nabla f_h^{\Delta t} \cdot \nabla \hat{f}_h \, dx + \int_\Omega (\Lambda_h^{\Delta t} f_h^{\Delta t}) \hat{f}_h \, dx = \langle y_T, \hat{f}_h \rangle \ \forall \hat{f}_h \in V_{0h}, \end{cases} \tag{1.425}$$

where, in (1.425), $\Lambda_h^{\Delta t}$ denotes the *linear operator* from V_{0h} into V_{0h} defined as follows

$$\Lambda_h^{\Delta t}\hat{f}_h = 2\hat{\varphi}_h^{N-1} - \hat{\varphi}_h^{N-2}, \tag{1.426}$$

where, to obtain $\hat{\varphi}_h^{N-1}$ and $\hat{\varphi}_h^{N-2}$, we solve for $n = N-1, \ldots, 1$, the well-posed discrete elliptic problem

$$\begin{cases} \hat{\psi}_h^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{\frac{3}{2}\hat{\psi}_h^n - 2\hat{\psi}_h^{n+1} + \frac{1}{2}\hat{\psi}_h^{n+2}}{\Delta t} z_h \, dx + a(z_h, \hat{\psi}_h^n) = 0 \;\forall z_h \in V_{0h}, \end{cases} \tag{1.427}$$

with

$$\hat{\psi}_h^N = 2\hat{f}_h, \quad \hat{\psi}_h^{N+1} = 4\hat{f}_h, \tag{1.428}$$

then, with $\hat{\varphi}_h^0 = 0$,

$$\begin{cases} \hat{\varphi}_h^1 \in V_{0h}, \\ \displaystyle\int_\Omega \frac{\hat{\varphi}_h^1 - \hat{\varphi}_h^0}{\Delta t} z_h \, dx + a(\tfrac{2}{3}\hat{\varphi}_h^1 + \tfrac{1}{3}\hat{\varphi}_h^0, z_h) = \tfrac{2}{3}\hat{\psi}_h^1(b) z_h(b) \;\forall z_h \in V_{0h}, \end{cases} \tag{1.429}$$

and, finally, for $n = 2, \ldots, N-1$,

$$\begin{cases} \hat{\varphi}_h^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{\frac{3}{2}\hat{\varphi}_h^{n-1} - 2\hat{\varphi}_h^{n-1} + \frac{1}{2}\hat{\psi}_h^{n-2}}{\Delta t} z_h \, dx + a(\hat{\varphi}_h^n, z_h) = \hat{\psi}_h^n(b) z_h(b) \;\forall z_h \in V_{0h}. \end{cases} \tag{1.430}$$

It follows from (1.426)–(1.430) that (with obvious notation)

$$\int_\Omega (\Lambda_h^{\Delta t} f_1) f_2 \, dx = \Delta t \sum_{n=1}^{N-1} \psi_1^n(b) \psi_2^n(b) \;\forall f_1, f_2 \in V_{0h},$$

i.e. operator $\Lambda_h^{\Delta t}$ is *symmetric* and *positive semi-definite*, which implies in turn that the approximate dual problem (1.425) has a *unique* solution.

Remark 1.51 The discrete problem (1.425) is actually the *dual problem* of the following *discrete control problem* (a variant of problem (1.389); see Section 1.10.6.2):

$$\min_{\mathbf{v} \in \mathbb{R}^{N-1}} J_h^{\Delta t}(\mathbf{v}), \tag{1.431}$$

where, in (1.431), we have $\mathbf{v} = \{v^n\}_{n=1}^{N-1}$ and

$$J_h^{\Delta t}(\mathbf{v}) = \tfrac{1}{2}\Delta t \sum_{n=1}^{N-1} |v^n|^2 + \tfrac{1}{2}k \int_\Omega |\nabla \Phi_h^N|^2 \, dx, \tag{1.432}$$

with Φ_h^N obtained from $\mathbf{v}$ via the solution of the following discrete parabolic problem

$$y_h^0 = 0, \tag{1.433}$$

$$\begin{cases} y_h^1 \in V_{0h}, \\ \int_\Omega \frac{y_h^1 - y_h^0}{\Delta t} + a(\tfrac{2}{3}y_h^1 + \tfrac{1}{3}y_h^0, z_h) = \tfrac{2}{3}v^1 z_h(b) \ \forall z_h \in V_{0h}, \end{cases} \tag{1.434}$$

then, for $n = 2, \ldots, N-1$, assuming that y_h^{n-1} is known we solve

$$\begin{cases} y_h^n \in V_{0h}, \\ \int_\Omega \frac{\frac{3}{2}y_h^n - 2y_h^{n-1} + \frac{1}{2}y_h^{n-2}}{\Delta t} z_h \,dx + a(y_h^n, z_h) = v^n z_h(b) \ \forall z_h \in V_{0h}, \end{cases} \tag{1.435}$$

and finally

$$\begin{cases} \Phi_h^N \in V_{0h}, \\ \int_\Omega \nabla\Phi_h^N \cdot \nabla z_h \,dx = \langle y_T - 2y_h^{N-1} + y_h^{N-2}, z_h\rangle \ \forall z_h \in V_{0h}. \end{cases} \tag{1.436}$$

Back to problem (1.425), it follows from the properties of operator $\Lambda_h^{\Delta t}$ that this problem can be solved by the following conjugate gradient algorithm (which is a discrete analogue of algorithm (1.373)–(1.385); see Section 1.10.5.3):

$$f_0 \in V_{0h} \textit{ is given;} \tag{1.437}$$

take

$$p_0^N = 2f_0, p_0^{N+1} = 4f_0, \tag{1.438}$$

and solve for $n = N-1, \ldots, 1$ *the following discrete elliptic problem*

$$\begin{cases} p_0^n \in V_{0h}, \\ \int_\Omega \frac{\frac{3}{2}p_0^n - 2p_0^{n+1} + \frac{1}{2}p_0^{n+2}}{\Delta t} z_h \,dx + a(z_h, p_0^n) = 0l \ \forall z_h \in V_{0h}. \end{cases} \tag{1.439}$$

Take now

$$y_0^0 = 0, \tag{1.440}$$

and solve

$$\begin{cases} y_0^1 \in V_{0h}, \\ \int_\Omega \frac{y_0^1 - y_0^0}{\Delta t} z_h \,dx + a(\tfrac{2}{3}y_0^1 + \tfrac{1}{3}y_0^0, z_h) = \tfrac{2}{3}p_0^1(b) z_h(b) \ \forall z_h \in V_{0h}; \end{cases} \tag{1.441}$$

solve next, for $n = 2, \dots, N-1$,

$$\begin{cases} y_0^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{\frac{3}{2}y_0^n - 2y_0^{n-1} + \frac{1}{2}y_0^{n-2}}{\Delta t} z_h \,dx + a(y_0^n, z_h) = p_0^n(b) z_h(b) \ \forall z_h \in V_{0h}. \end{cases} \tag{1.442}$$

Solve, next

$$\begin{cases} g_0 \in V_{0h}, \\ \displaystyle\int_\Omega \nabla g_0 \cdot \nabla \hat{f} \,dx = k^{-1} \int_\Omega \nabla f_0 \cdot \nabla \hat{f} \,dx \\ \qquad + \langle 2y_0^{N-1} - y_0^{N-2} - y_T, \hat{f} \rangle \ \forall \hat{f} \in V_{0h}, \end{cases} \tag{1.443}$$

and set

$$w_0 = g_0. \quad \square \tag{1.444}$$

Then for $m \geq 0$, *assuming that* f_m, g_m, w_m *are known compute* f_{m+1}, g_{m+1}, w_{m+1} *as follows.*

Take

$$\bar{p}_m^N = 2w_m, \bar{p}_m^{N+1} = 4w_m \tag{1.445}$$

and solve for $n = N-1, \dots, 1$

$$\begin{cases} \bar{p}_m^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{\frac{3}{2}\bar{p}_m^n - 2\bar{p}_m^{n+1} + \frac{1}{2}p_m^{n+2}}{\Delta t} z_h \,dx + a(z_h, \bar{p}_m^n) = 0 \ \forall z_h \in V_{0h}. \end{cases} \tag{1.446}$$

Take

$$\bar{y}_m^0 = 0; \tag{1.447}$$

solve

$$\begin{cases} \bar{y}_m^1 \in V_{0h}, \\ \displaystyle\int_\Omega \frac{\bar{y}_m^1 - \bar{y}_m^0}{\Delta t} z_h \,dx + a(\tfrac{2}{3}\bar{y}_m^1 + \tfrac{1}{3}\bar{y}_m^0, z_h) = \tfrac{2}{3}\bar{p}_m^1(b) z_h(b) \ \forall z_h \in V_{0h}, \end{cases} \tag{1.448}$$

and then for $n = 2, \dots, N-1$

$$\begin{cases} \bar{y}_m^n \in V_{0h}, \\ \displaystyle\int_\Omega \frac{\frac{3}{2}\bar{y}_m^n - 2\bar{y}_m^{n-1} + \frac{1}{2}\bar{y}_m^{n-2}}{\Delta t} z_h \,dx + a(\bar{y}_m^n, z_h) = \bar{p}_m^n(b) z_h(b) \ \forall z_h \in V_{0h}. \end{cases} \tag{1.449}$$

Solve next

$$\begin{cases} \bar{g}_m \in V_{0h}, \\ \displaystyle\int_\Omega \nabla \bar{g}_m \cdot \nabla \hat{f}\,\mathrm{d}x = k^{-1} \int_\Omega \nabla w_m \cdot \nabla \hat{f}\,\mathrm{d}x \\ \qquad\qquad + \displaystyle\int_\Omega (2\bar{y}_m^{N-1} - \bar{y}_m^{N-2})\hat{f}\,\mathrm{d}x \ \forall \hat{f} \in V_{0h}, \end{cases} \tag{1.450}$$

and compute

$$\rho_m = \int_\Omega |\nabla g_m|^2\,\mathrm{d}x \Big/ \int_\Omega \nabla \bar{g}_m \cdot \nabla w_m\,\mathrm{d}x; \tag{1.451}$$

then update f_m *and* g_m *by*

$$f_{m+1} = f_m - \rho_m w_m, \tag{1.452}$$

$$g_{m+1} = g_m - \rho_m \bar{g}_m, \tag{1.453}$$

respectively. If $\|g_{m+1}\|_{H_0^1(\Omega)}/\|g_0\|_{H_0^1(\Omega)} \le \epsilon$, *take* $f_h^{\Delta t} = f_{m+1}$, *else compute*

$$\gamma_m = \int_\Omega |\nabla g_{m+1}|^2\,\mathrm{d}x \Big/ \int_\Omega |\nabla g_m|^2\,\mathrm{d}x \tag{1.454}$$

and update w_m *by*

$$w_{m+1} = g_{m+1} + \gamma_m w_m. \quad \square \tag{1.455}$$

Do $m = m + 1$ *and go to* (1.445).

Algorithm (1.437)–(1.455) is fairly easy to implement. It essentially requires *elliptic/finite element solvers* to compute the solutions to problems (1.439), (1.441)–(1.443), (1.446), (1.448)–(1.450); such solvers are easily available.

1.10.7. Numerical experiments

1.10.7.1. Generalities. Synopsis. In order to illustrate the results and methods from Sections 1.10.1 to 1.10.6 we shall discuss in this section the solution of some *pointwise control problems*; these problems will be particular cases and variants of the *penalized* problem (1.351). We suppose for simplicity that $d = 1$ (i.e. $\Omega \subset \mathbb{R}$); it follows then from Remark 1.45 that the solution of (1.317) satisfies

$$y \in C^0([0,T]; L^2(\Omega)),$$

which implies that, in (1.351), it makes sense to replace $\|y(T) - y_T\|_{-1}$ by $\|y(T) - y_T\|_{L^2(\Omega)}$. Also, for some of the test problems we shall replace $\frac{1}{2}\int_0^T |v|^2\,\mathrm{d}t$ by $(1/s)\int_0^T |v|^s\,\mathrm{d}t$, with $s > 2$, including some *very large* values of s for which the optimal control u (in fact, its discrete analogue) clearly has a *bang–bang behaviour*; this is expected from Section 1.7.

1.10.7.2. First test problems. What we have considered here is a family of test problems parametrized by T, y_T, k and by the 'support' b of the pointwise control. These test problems can be formulated as follows.

$$\min_{v \in L^2(0,T)} J_k(v), \tag{1.456}$$

where

$$J_k(v) = \tfrac{1}{2}\int_0^T v^2\,\mathrm{d}t + \frac{k}{2}\int_0^1 |y(T) - y_T|^2\,\mathrm{d}x, \tag{1.457}$$

with y the solution of the following *diffusion problem*

$$\frac{\partial y}{\partial t} - \nu\frac{\partial^2 y}{\partial x^2} = v(t)\delta(x-b) \text{ in } (0,1)\times(0,T), \tag{1.458}$$

$$y(0,t) = y(1,t) = 0 \text{ on } (0,T), \tag{1.459}$$

$$y(0) = 0. \tag{1.460}$$

In equation (1.458) we have $\nu > 0$ and $b \in (0,1)$. We clearly have $\Omega = (0,1)$.

We have considered, first, test problems where the *target function* y_T is *even* with respect to the variable $x - \frac{1}{2}$. These target functions are given by

$$y_T(x) = 4x(1-x), \tag{1.461}$$

$$y_T(x) = \begin{cases} 8(x-\frac{1}{4}) & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ 8(\frac{3}{4}-x) & \text{if } \frac{1}{2} \le x \le \frac{3}{4}, \\ 0 & \text{elsewhere on } (0,1), \end{cases} \tag{1.462}$$

$$y_T(x) = \begin{cases} 1 & \text{if } \frac{1}{4} \le x \le \frac{3}{4}, \\ 0 & \text{elsewhere on } (0,1), \end{cases} \tag{1.463}$$

respectively. We have taken $\nu = \frac{1}{10}$ in equation (1.458), and $T = 3$ for all the three target functions we have given above. The continuous problem (1.456)–(1.460) has been approximated using the methods described in Sections 1.10.6.2 to 1.10.6.4 (i.e. we have solved *directly* the control problems, taking into account the fact that for these test problems we use a penalty term associated with the L^2 norm, instead of the H^{-1} norm used in the general case). The *time discretization* has been obtained using the *backward Euler scheme* described in Section 1.10.6.2 with $\Delta t = 10^{-2}$, while the space discretization was obtained using a *uniform* mesh on (0,1) with $h = 10^{-2}$. The discrete control problems have been solved by a variant of the *conjugate gradient* algorithm (1.406)–(1.424); we have taken $\mathbf{u}_0 = \mathbf{0}$ as *initializer* for the above algorithm and $\|\mathbf{g}_m\|_{\Delta t}/\|\mathbf{g}_0\|_{\Delta t} \le 10^{-6}$ as the *stopping criterion* (for those cases for which this criterion could not be reached sufficiently quickly, we stopped iterating after a fixed number of iterations (300 or 500,

Table 1. *Summary of numerical results (target function defined by (1.461); $T = 3,\ h = \Delta t = 10^{-2}$).*

b	k	Number of iterations	$\|u^*\|_{L^2(0,T)}$	$\frac{\|y^*(T)-y_T\|_{L^2(0,1)}}{\|y_T\|_{L^2(0,1)}}$
$\sqrt{2}/3$	10^2	95	0.923	6×10^{-2}
	10^3	> 300	1.14	2.3×10^{-2}
	10^4	> 300	1.28	1.4×10^{-2}
$1/2$	10^2	93	0.909	5.5×10^{-2}
	10^3	> 300	1.09	2.1×10^{-2}
	10^4	> 300	1.20	1.3×10^{-2}
$\pi/6$	10^2	95	0.918	5.9×10^{-2}
	10^3	> 300	1.12	2.3×10^{-2}
	10^4	> 300	1.26	1.4×10^{-2}

depending on the test problem)). The corresponding numerical results have been summarized in Tables 1 to 3, where u^* and $y^*(T)$ denote the computed optimal control and the corresponding final state, respectively.

In Figures 1 to 9 we have visualized, for $k = 10^4$, the computed optimal control and compared the corresponding computed value of $y(T)$ (i.e. $y^*(T)$) to the target function y_T.

The above results deserve several comments:

(i) Since operator $A = -\nu\, \mathrm{d}^2/\mathrm{d}x^2$ is *self-adjoint* for the *homogeneous* Dirichlet boundary conditions we can apply the controllability results of Section 1.10.2. The eigenfunctions of operator A, i.e. the solutions of

$$-\nu \frac{\mathrm{d}^2}{\mathrm{d}x^2} w_j = \lambda_j w_j \text{ on } (0,1), \quad w_j(0) = w_j(1) = 0, \quad w_j \neq 0,$$

are clearly given by

$$w_j(x) = \sin j\pi x, \quad j = 1, 2, \ldots,$$

the corresponding *spectrum* being $\{\nu\pi^2 j^2\}_{j=1}^{+\infty}$. Since each eigenvalue is *simple*, b will be *strategic* if

$$\sin j\pi b \neq 0 \ \forall j = 1, 2, \ldots,$$

i.e. if

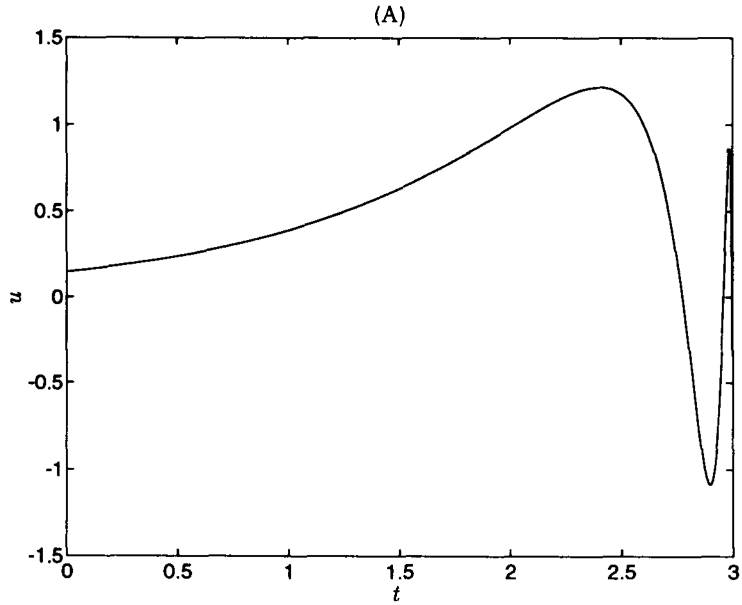
$$b \notin (0,1) \cap \mathbb{Q} \tag{1.464}$$

Table 2. *Summary of numerical results (target function defined by (1.462); $T = 3,\ h = \Delta t = 10^{-2}$).*

b	k	Number of iterations	$\|u^*\|_{L^2(0,T)}$	$\frac{\|y^*(T)-y_T\|_{L^2(0,1)}}{\|y_T\|_{L^2(0,1)}}$
$\sqrt{2}/3$	10^2	157	1.23	2.2×10^{-1}
	10^3	> 500	1.93	1.9×10^{-1}
	10^4	> 500	3.01	1.8×10^{-1}
$1/2$	10^2	113	1.27	1.1×10^{-1}
	10^3	> 500	1.74	6.3×10^{-2}
	10^4	> 500	2.03	5.6×10^{-2}
$\pi/6$	10^2	137	1.24	1.9×10^{-1}
	10^3	> 500	1.82	1.6×10^{-1}
	10^4	> 500	2.72	1.5×10^{-1}

Table 3. *Summary of numerical results (target function defined by (1.463); $T = 3,\ h = \Delta t = 10^{-2}$).*

b	k	Number of iterations	$\|u^*\|_{L^2(0,T)}$	$\frac{\|y^*(T)-y_T\|_{L^2(0,1)}}{\|y_T\|_{L^2(0,1)}}$
$\sqrt{2}/3$	10^2	279	0.94	3.47×10^{-1}
	10^3	> 500	2.2	3.13×10^{-1}
	10^4	> 500	3.0	3×10^{-1}
$1/2$	10^2	317	1.0	3.25×10^{-1}
	10^3	> 500	2.6	2.79×10^{-1}
	10^4	> 500	3.6	2.64×10^{-1}
$\pi/6$	10^2	291	0.96	3.4×10^{-1}
	10^3	> 500	2.3	3×10^{-1}
	10^4	> 500	3.2	2.9×10^{-1}



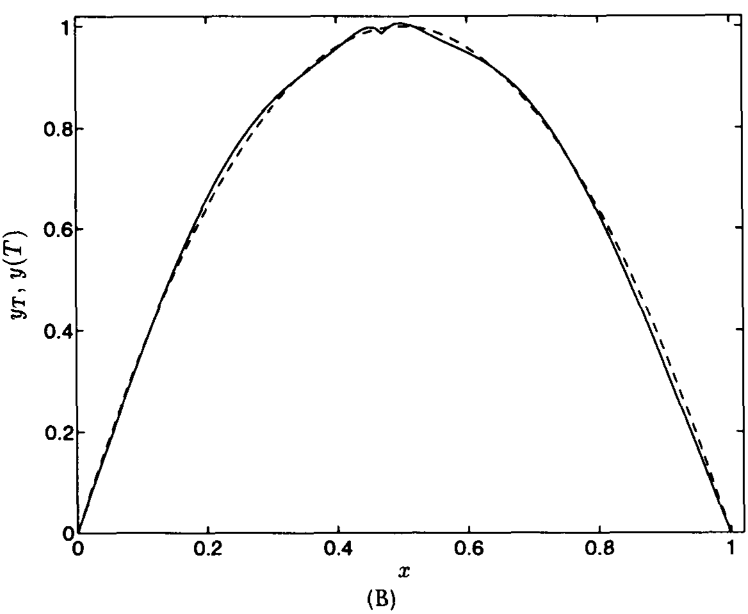


Fig. 1. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.461): $T = 3$, $b = \sqrt{2}/3$, $k = 10^4$, $h = \Delta t = 10^{-2}$).

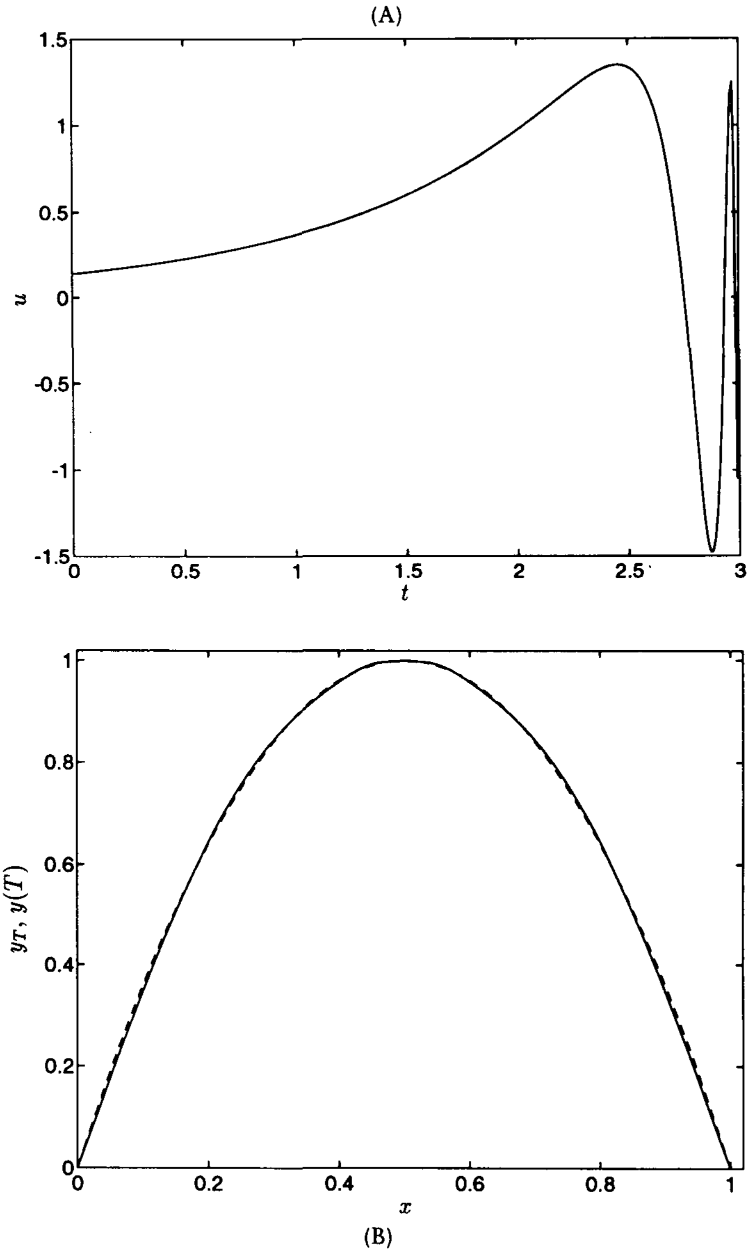


Fig. 2. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.461): $T = 3$, $b = 1/2$, $k = 10^4$, $h = \Delta t = 10^{-2}$).

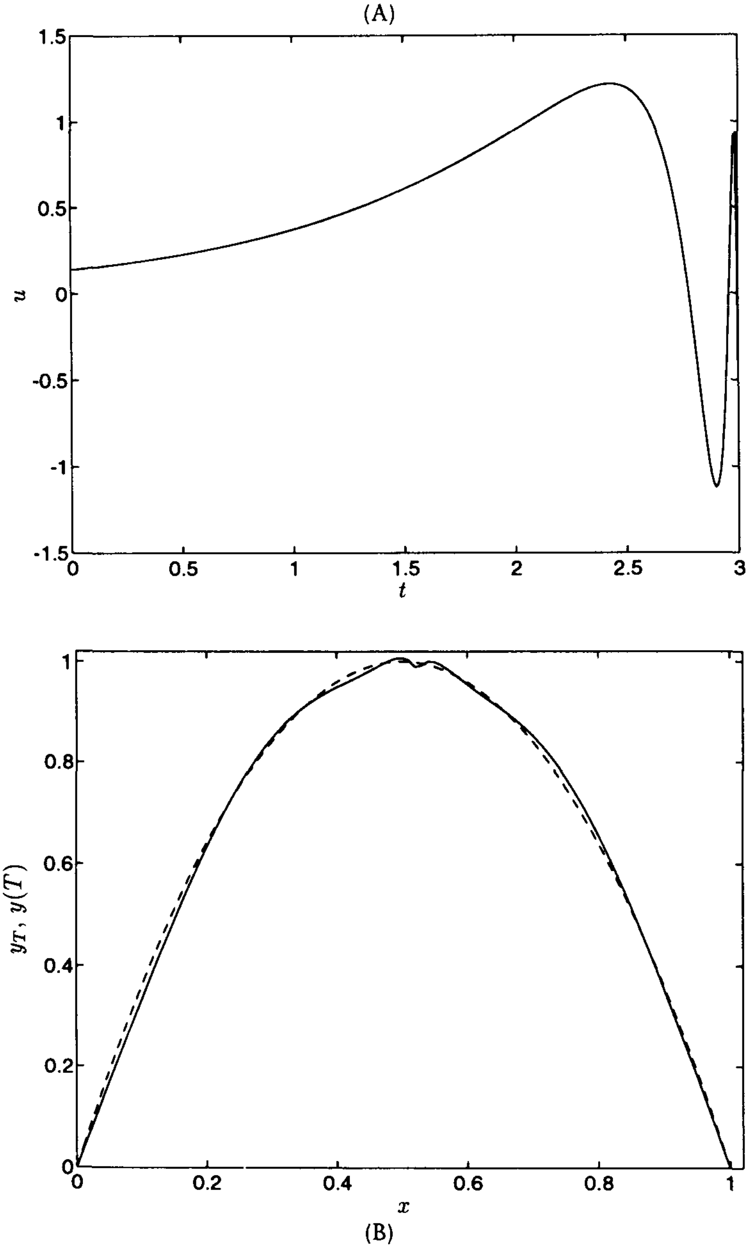


Fig. 3. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.461): $T = 3$, $b = \pi/6$, $k = 10^4$, $h = \Delta t = 10^{-2}$).

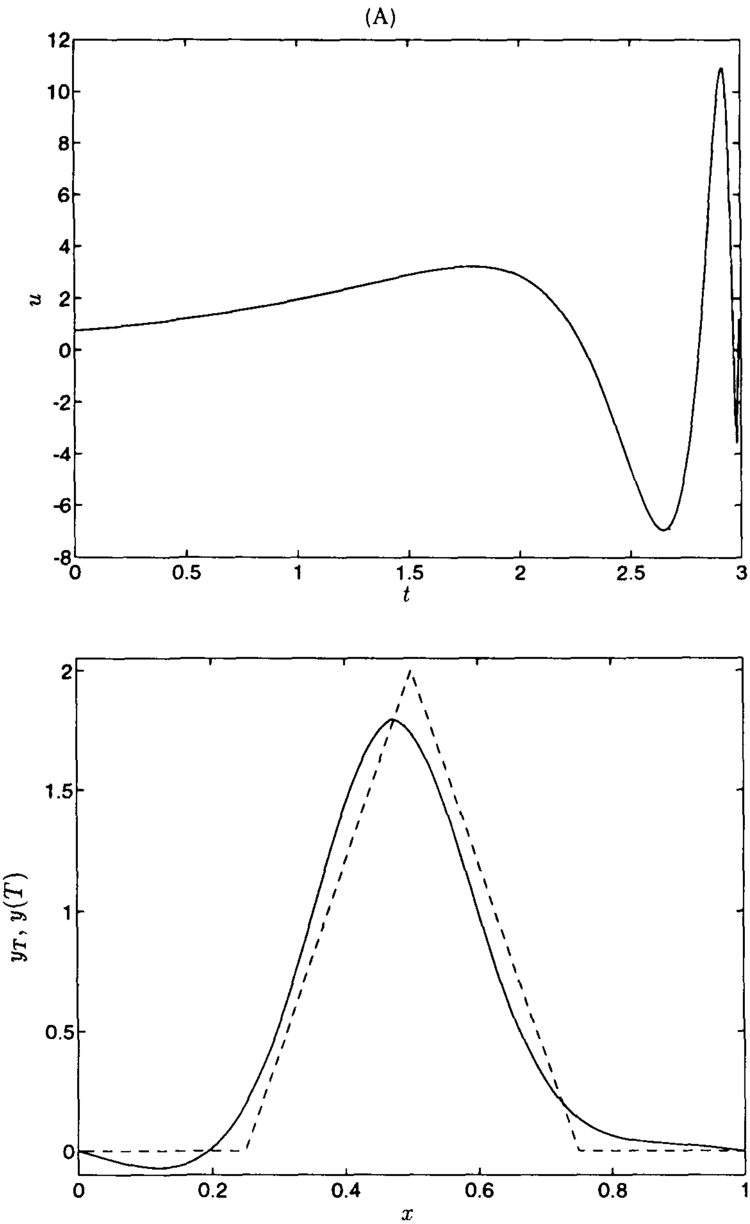


Fig. 4. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.462): $T = 3$, $b = \sqrt{2}/3$, $k = 10^4$, $h = \Delta t = 10^{-2}$).

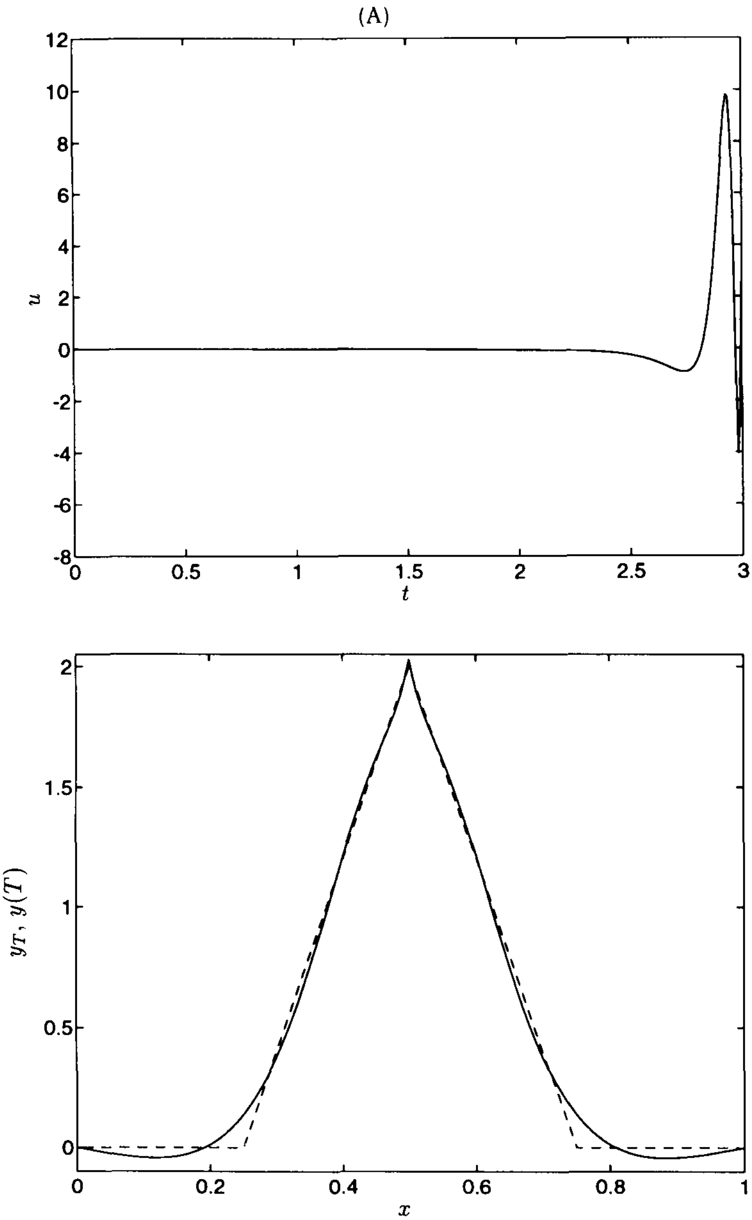


Fig. 5. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.462): $T = 3$, $b = 1/2$, $k = 10^4$, $h = \Delta t = 10^{-2}$).

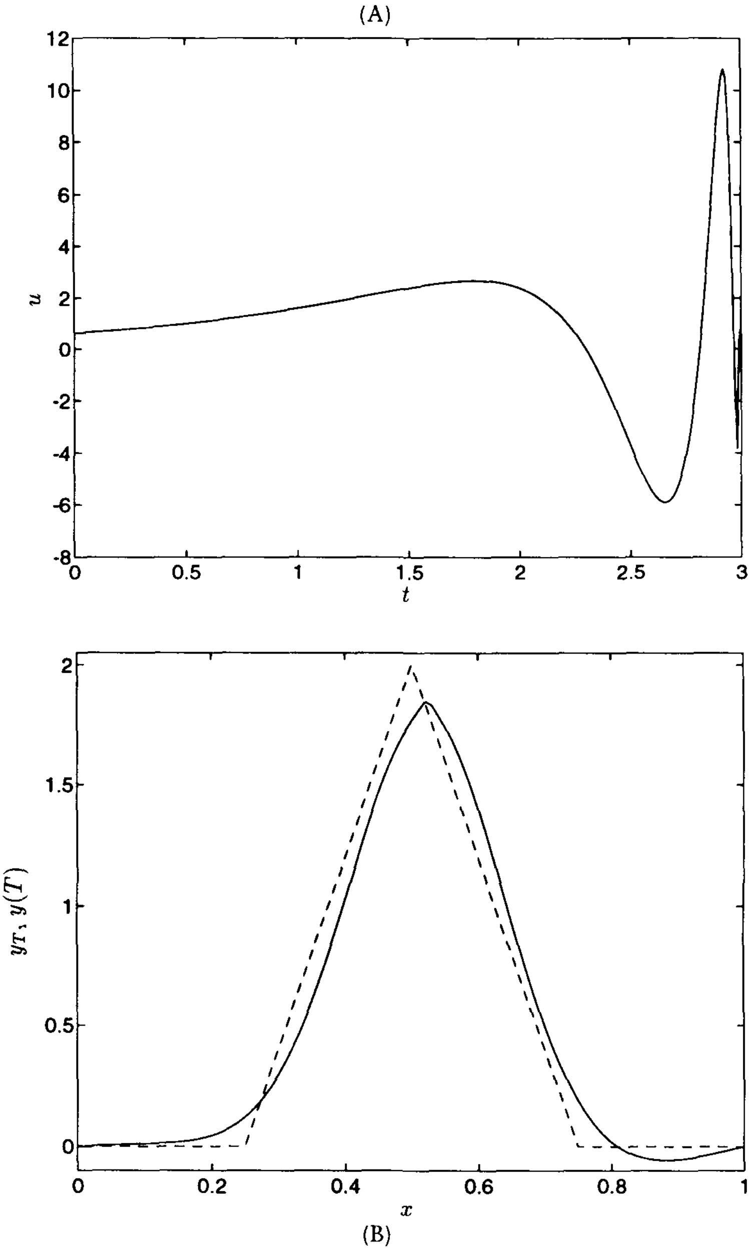


Fig. 6. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.462): $T = 3$, $b = \pi/6$, $k = 10^4$, $h = \Delta t = 10^{-2}$).

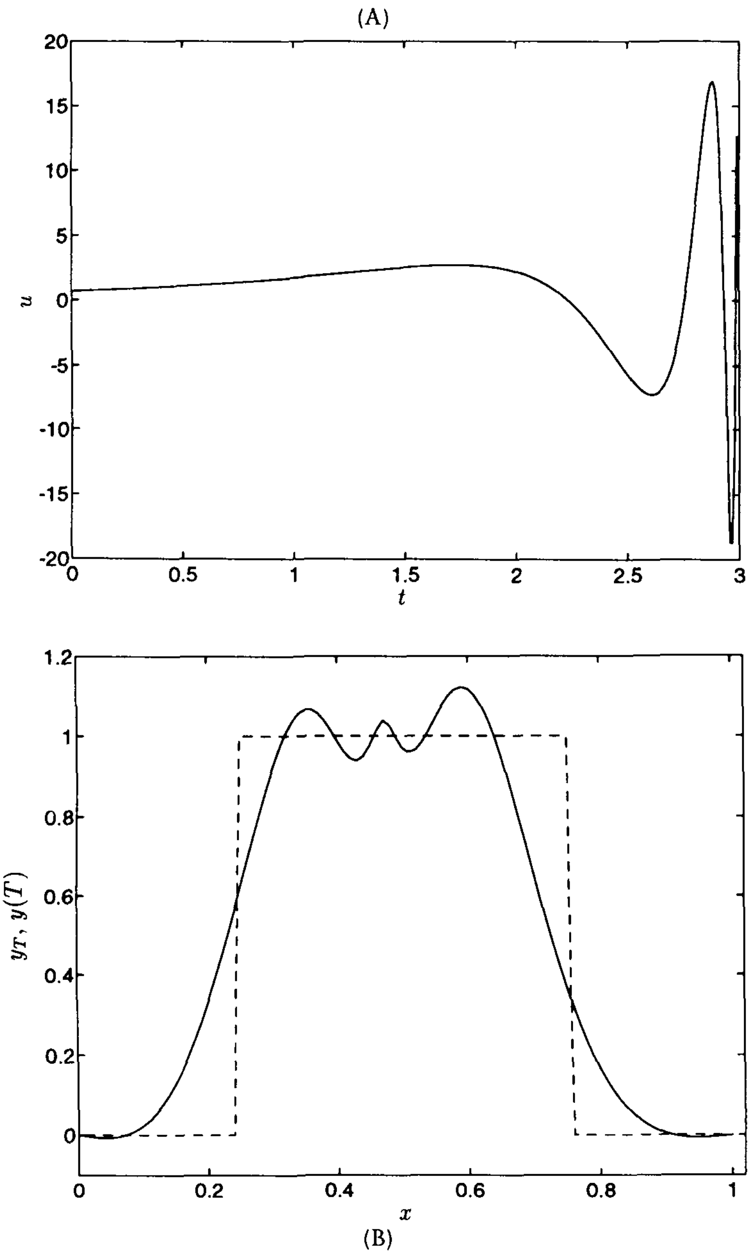


Fig. 7. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.463): $T = 3$, $b = \sqrt{2}/3$, $k = 10^4$, $h = \Delta t = 10^{-2}$.)

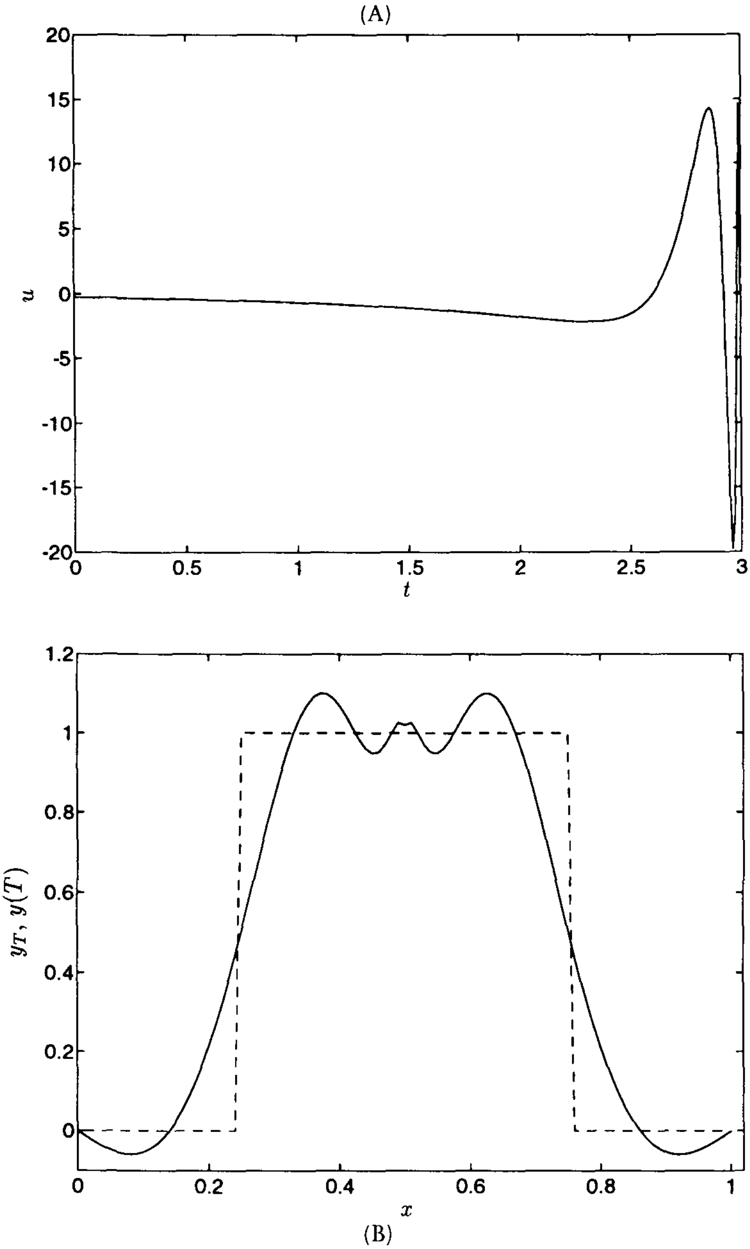
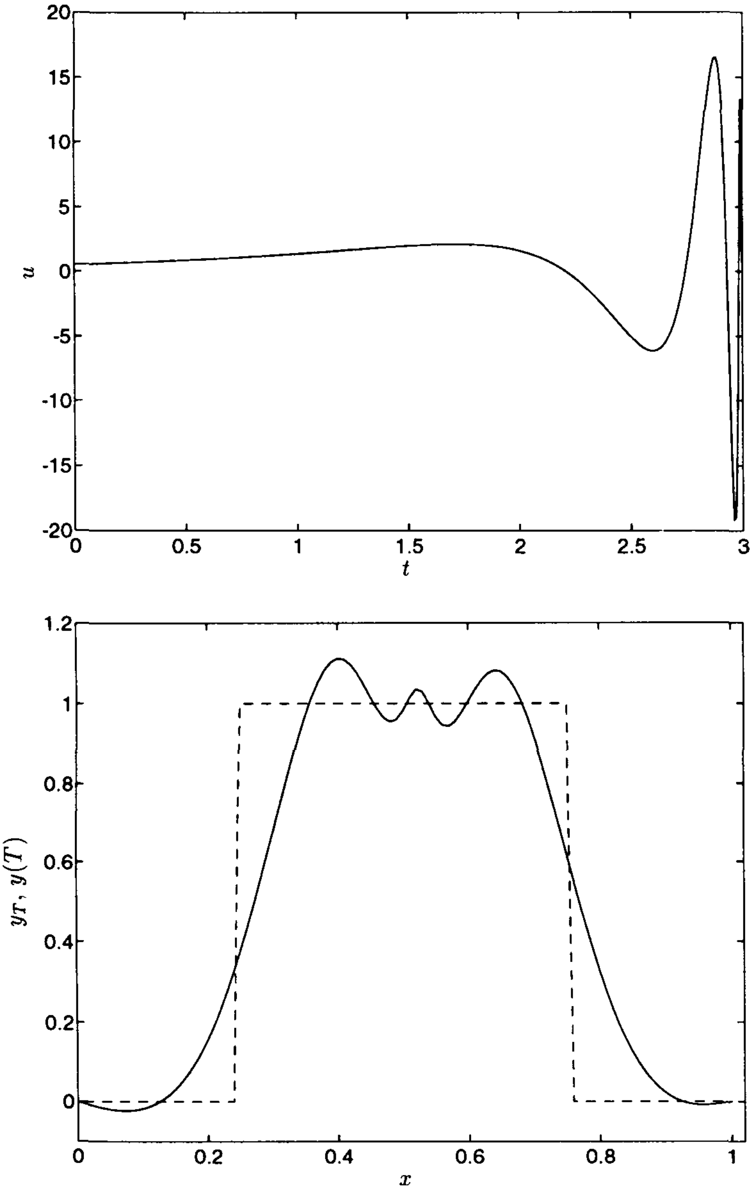


Fig. 8. Variation of optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.463): $T = 3$, $b = 1/2$, $k = 10^4$, $h = \Delta t = 10^{-2}$).

(A)



(B)

Fig. 9. Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.463): $T = 3$, $b = \pi/6$, $k = 10^4$, $h = \Delta t = 10^{-2}$).

where, in (1.464), $\mathbb{Q}$ is the field of the *rational* real numbers. Clearly, $b = \sqrt{2}/3$ and $b = \pi/6$ being nonrational are strategic. On the other hand, $b = \frac{1}{2}$ is far from being strategic since $\sin j\pi/2 = 0$ for any *even integer* j; indeed, $b = \frac{1}{2}$ is *generically* the worst choice which can be made. However, if one takes $b = \frac{1}{2}$ the solution y of problem (1.458)–(1.460) satisfies

$$\forall t \in [0,T], y(t) \text{ is an even function of } x - \tfrac{1}{2}; \tag{1.465}$$

property (1.465) implies that the coefficients of w_j in the Fourier expansion of y are zero for j even. This property implies in turn that $b = \frac{1}{2}$ is strategic if y_T is also an even function of $x - \frac{1}{2}$; this is precisely the case for the target functions defined by (1.461)–(1.463). Actually, for target functions y_T which are even with respect to $x - \frac{1}{2}$, $b = \frac{1}{2}$ is the best strategic point; this appears clearly in Tables 1 to 3 where the smallest *control norms* and *controllability errors* are obtained for $b = \frac{1}{2}$. In Section 1.10.7.3 we shall consider target functions which are not even with respect to $x - \frac{1}{2}$; $b = \frac{1}{2}$ will not be strategic at all for these test problems.

(ii) A *digital computer* 'knows' only rational numbers; this means that for the particular test problems considered in this section, strictly speaking, there is no strategic point for pointwise control. However, if b is the computer approximation of a nonrational number, the integers j such that bj are also integers are *very large.* This last property implies that, unless h is extremely small and y_T quite pathological (i.e. its Fourier coefficients do not converge quickly to zero as $j \to +\infty$), such a b is strategic in practice.

(iii) The discrete control problems approximating (1.456) are equivalent to linear systems associated with an $N \times N$ symmetric and positive-definite matrix (we recall that $N = T/\Delta t$). These problems can be solved, therefore, by conjugate gradient algorithms. From the classical properties of conjugate gradient methods (see, e.g., Ciarlet (1989), Golub and Van Loan (1989)), we expect convergence in N iterations at most. Looking at Tables 1 to 3 we observe that for k sufficiently large this finite termination property does not hold. The main reason for this behaviour is that these discrete control problems are *badly conditioned* for large values of k, implying high sensitivity to round-off errors and, consequently, loss of the finite termination property.

An alternative to conjugate gradient methods is to construct the matrix and right-hand side of the equivalent linear system and to solve it by Cholesky's method. Let us briefly evaluate the cost of constructing the matrix and the right-hand side of this linear system. It follows from Section 1.10.6.3 that to construct the matrix (respectively the right-hand side) we need to solve N (respectively 1) discrete forward parabolic problems and then N (respectively 1) discrete backward parabolic problems, implying a total of $2(N + 1)$ parabolic problems. If one modifies y_T, with everything else staying the same, we only have to compute the corresponding new right-hand side at the cost of solving two discrete parabolic problems.

Table 4. *Summary of numerical results (target function defined by (1.466); $T = 3$, $h = \Delta t = 10^{-2}$).*

b	k	$\|u^*\|_{L^2(0,T)}$	$\frac{\|y^*(T)-y_T\|_{L^2(0,T)}}{\|y_T\|_{L^2(0,T)}}$
$\sqrt{2}/3$	10^4	42.4	1.5×10^{-1}
	10^5	73.6	4×10^{-2}
$1/2$	10^4	4.4	3.5×10^{-1}
	10^5	5.46	3.5×10^{-1}
$\pi/6$	10^4	40.4	1.8×10^{-1}
	10^5	84.5	5.6×10^{-2}

1.10.7.3. Further test problems The test problems in this section are still defined by (1.456)–(1.460) the main difference being that the target functions y_T are not even with respect to the variable $x - \frac{1}{2}$. Indeed, the two target functions considered here are defined by

$$y_T(x) = \tfrac{27}{4}x^2(1 - x), \tag{1.466}$$

and

$$y_T(x) = \begin{cases} 0 & \text{on } [0, \tfrac{1}{2}], \\ 8(x - \tfrac{1}{2}) & \text{on } [\tfrac{1}{2}, \tfrac{3}{4}], \\ 8(1 - x) & \text{on } [\tfrac{3}{4}, 1]; \end{cases} \tag{1.467}$$

we have taken $T = 3$ for both target functions. The approximation and solution methods being those of Section 1.10.7.2, still with $\nu = \frac{1}{10}$, $h = \Delta t = 10^{-2}$, we have obtained the results summarized in the Tables 4 and 5 and Figures 10–15 below (the notation is the same as in Section 1.10.7.2).

These results clearly show that $b = \frac{1}{2}$ is not strategic for the test problems considered here; this was expected since none of the functions y_T is even with respect to $x - \frac{1}{2}$. On the other hand, 'small' *irrational* shifts, either to the right or to the left of $\frac{1}{2}$, produce strategic values of b. The other comments made in Section 1.10.7.2 still hold for the examples considered here.

1.10.7.4. Test problems for nonquadratic cost functions. Motivated by Section 1.9 we have been considering pointwise control problems defined by

$$\min_{v \in L^s(0,T)} \left[\tfrac{1}{2}\|v\|^2_{L^s(0,T)} + \tfrac{1}{2}k\|y(T) - y_T\|^2_{L^2(0,1)}\right], \tag{1.468}$$

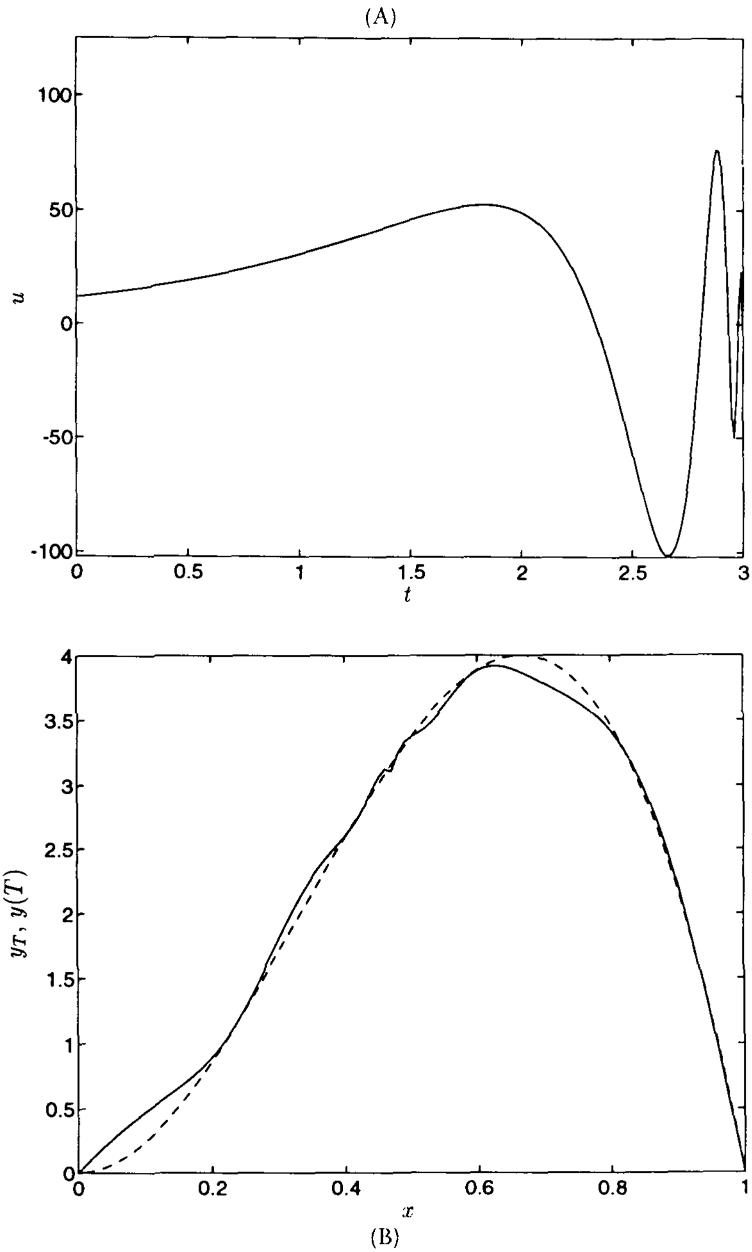
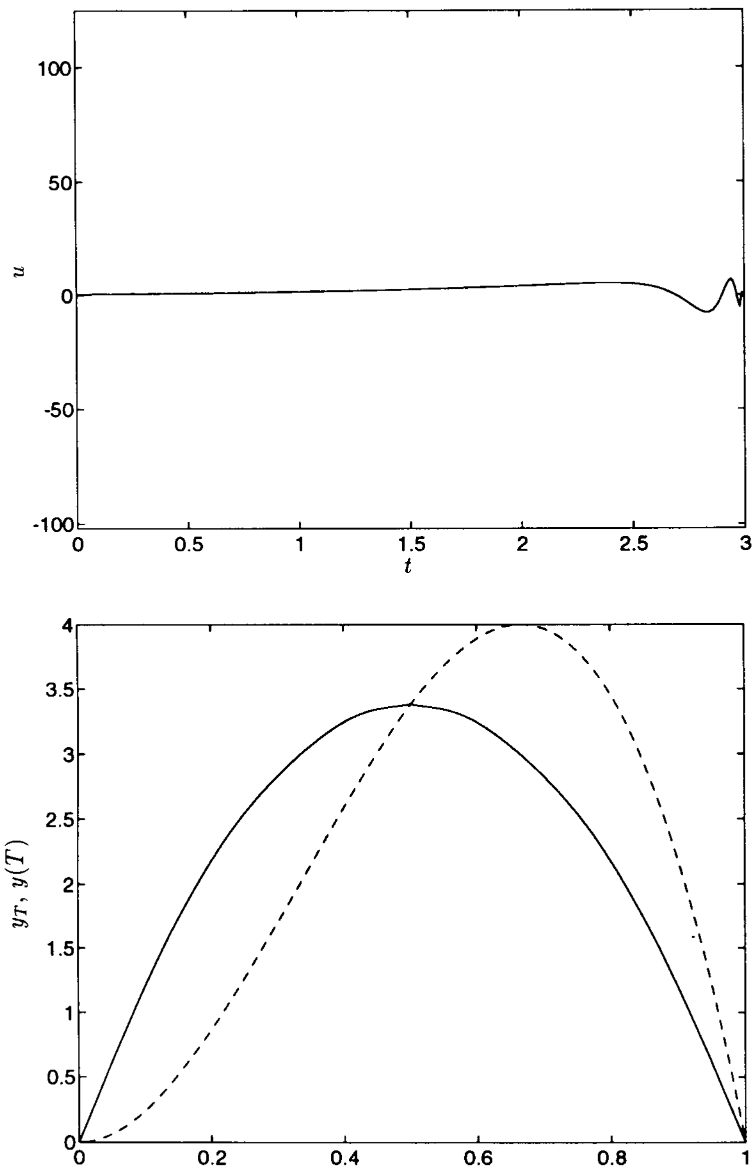


Fig. 10. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466): $T = 3$, $b = \sqrt{2}/3$, $k = 10^5$, $h = \Delta t = 10^{-2}$).

(A)



(B)

Fig. 11. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466): $T = 3$, $b = \frac{1}{2}$, $k = 10^5$, $h = \Delta t = 10^{-2}$).

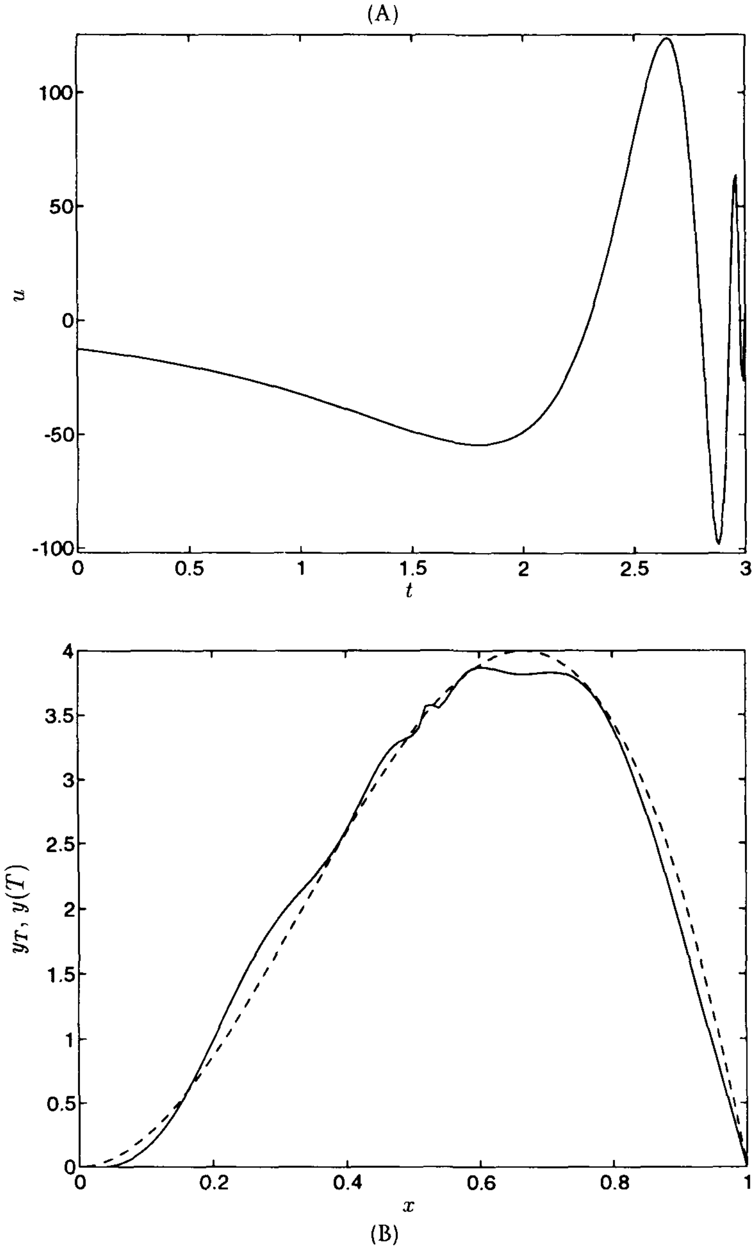


Fig. 12. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466); $T = 3$, $b = \pi/6$, $k = 10^5$, $h = \Delta t = 10^{-2}$).

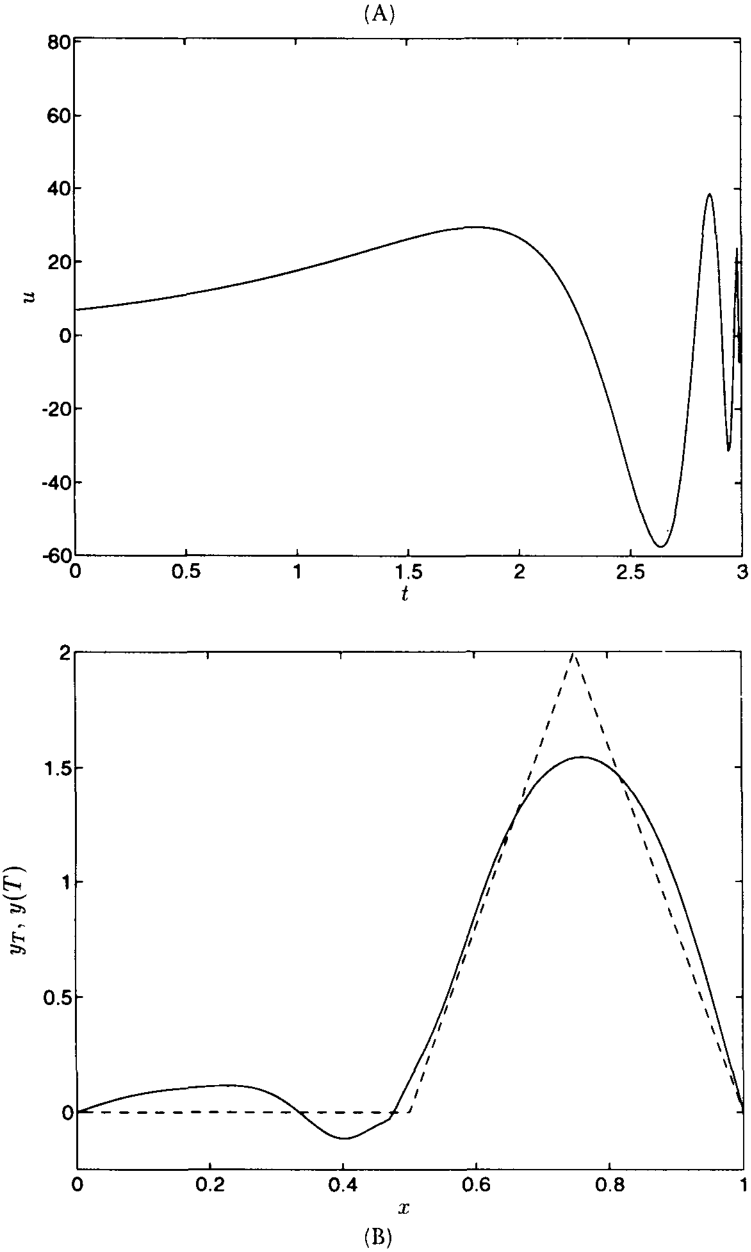


Fig. 13. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.467): $T = 3$, $b = \sqrt{2}/3$, $k = 10^5$, $h = \Delta t = 10^{-2}$).

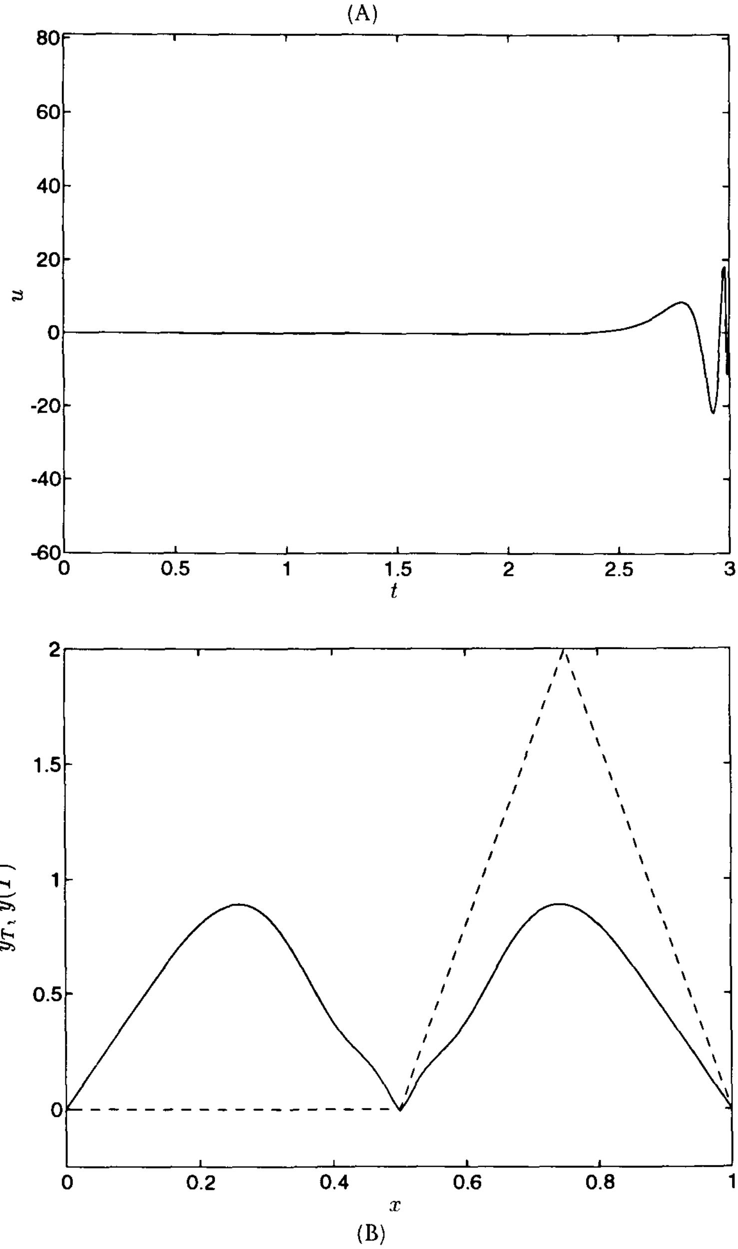


Fig. 14. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.467): $T = 3$, $b = \frac{1}{2}$, $k = 10^5$, $h = \Delta t = 10^{-2}$).

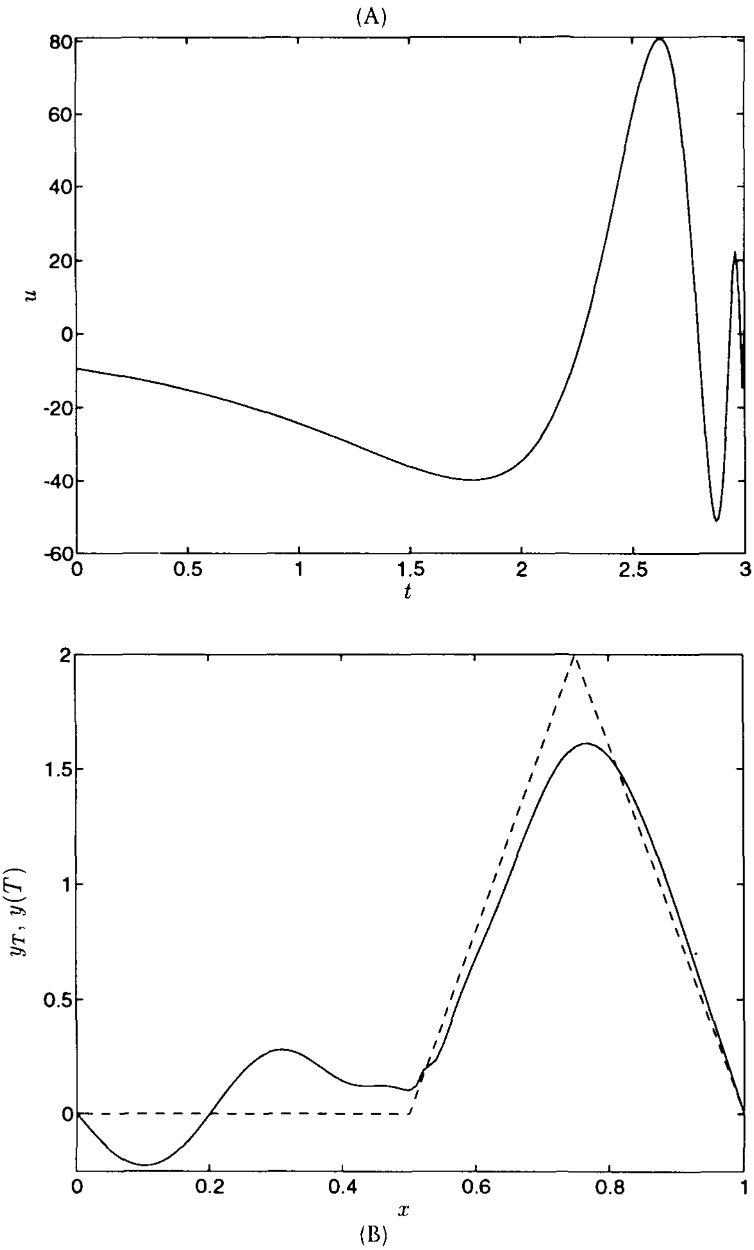


Fig. 15. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.467): $T = 3$, $b = \pi/6$, $k = 10^5$, $h = \Delta t = 10^{-2}$).

Table 5. *Summary of numerical results (target function defined by (1.467); $T = 3,\ h = \Delta t = 10^{-2}$).*

b	k	$\|u^*\|_{L^2(0,T)}$	$\frac{\|y^*(T)-y_T\|_{L^2(0,T)}}{\|y_T\|_{L^2(0,T)}}$
$\sqrt{2}/3$	10^3	6.53	5.2×10^{-1}
	10^4	21.8	3×10^{-1}
	10^5	41.6	1.6×10^{-1}
$1/2$	10^3	2.05	7.1×10^{-1}
	10^4	2.67	7.1×10^{-1}
	10^5	6.26	7.1×10^{-1}
$\pi/6$	10^3	5.99	6.7×10^{-1}
	10^4	27.5	4.2×10^{-1}
	10^5	57.6	2×10^{-1}

with y still defined from v by (1.458)–(1.460), and s 'large'. It seems, unfortunately, that for $s > 2$, problem (1.468) is poorly conditioned implying that the various iterative methods we used to solve it (conjugate gradient, Newton and quasi-Newton methods) have failed to converge (or even worse, have stuck on some wrong solution). From these facts it is quite natural to consider the variation of problem (1.468) defined by

$$\min_{v \in L^s(0,T)} \left[\frac{1}{s} \int_0^T |v(t)|^s \, dt + \tfrac{1}{2} k \|y(T) - y_T\|^2_{L^2(0,1)} \right], \tag{1.469}$$

with y defined from v as above. The cost function in (1.469) has better differentiability properties than the one in (1.468).

Let us denote by u the solution of (1.469); assuming that b in (1.458) is strategic we can expect that for s fixed $y(u;T)$ will get closer to y_T as k increases. If, on the other hand, k is fixed we can expect the distance from $y(u;T)$ to y_T to increase with s, since in that case the relative importance of the term $s^{-1}\int_0^T |v|^s \, dt$ in the cost function increases with s. These predictions are fully confirmed by the numerical experiments whose results are shown below. For these experiments we have used essentially the same approximation methods as in Sections 1.10.7.2 and 1.10.7.3, with $h = \Delta t = 10^{-2}$, and taken $b = \sqrt{2}/3$, $T = 3$, $\nu = \frac{1}{10}$ and y_T defined by (1.466). The discrete control problems have been solved by *quasi-Newton's methods* à la BFGS, like those discussed, for example, in the classical text book by Dennis and Schnabel (1983) (see also Nocedal (1992)); for the problems considered here, these methods appear to be much more efficient than conjugate gradient methods.

On Figures 16 to 21 we have – for $k = 10^7$ and $s = 2, 4, 6, 10, 20, 30$ – visualized the computed optimal control u^* and compared the corresponding final state $y^*(T)$ with the target y_T. From these figures, we clearly see that the distance of $y^*(T)$ to y_T increases with s; we also see the *bang–bang* character of the optimal control for large values of s.

Finally, on Figures 22 to 25 we have shown some of the results obtained for large values of s and very large values of k; comparing these with Figures 16 to 21 we observe that if for a given s we increase k, then $y^*(T)$ gets closer to $y(T)$ and $\|u^*\|_{L^2(0,T)}$ increases, which makes sense. We observe again that for very large values of s the optimal control is very close to bang–bang.

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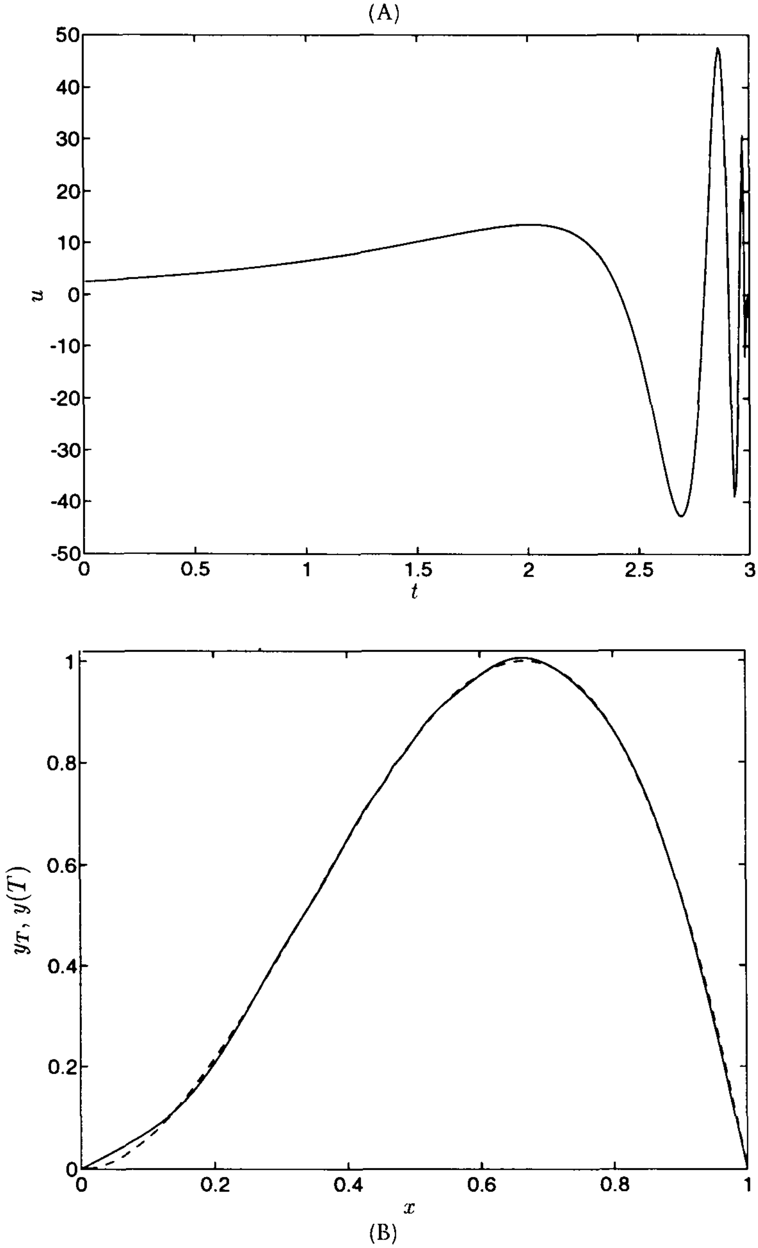


Fig. 16. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466): $s = 2$, $T = 3$, $b = \sqrt{2}/3$, $k = 10^7$, $h = \Delta t = 10^{-2}$).

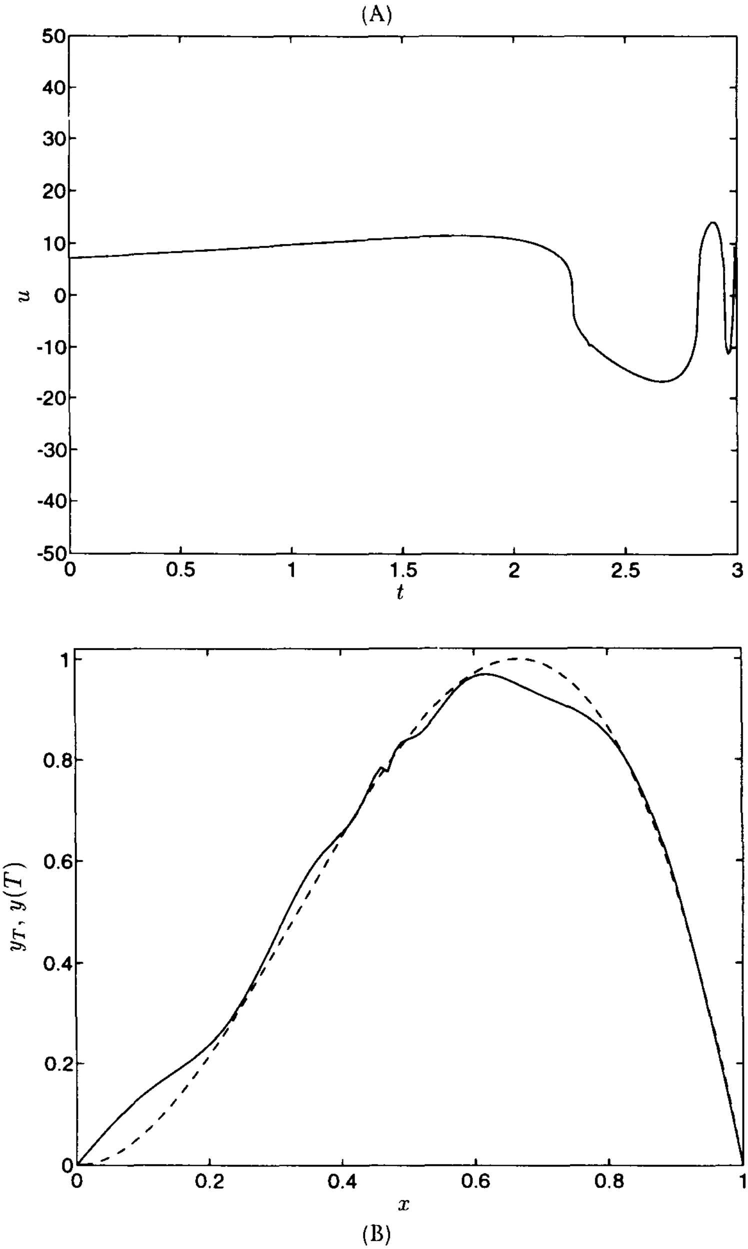


Fig. 17. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466): $s = 4$, $T = 3$, $b = \sqrt{2}/3$, $k = 10^7$, $h = \Delta t = 10^{-2}$).

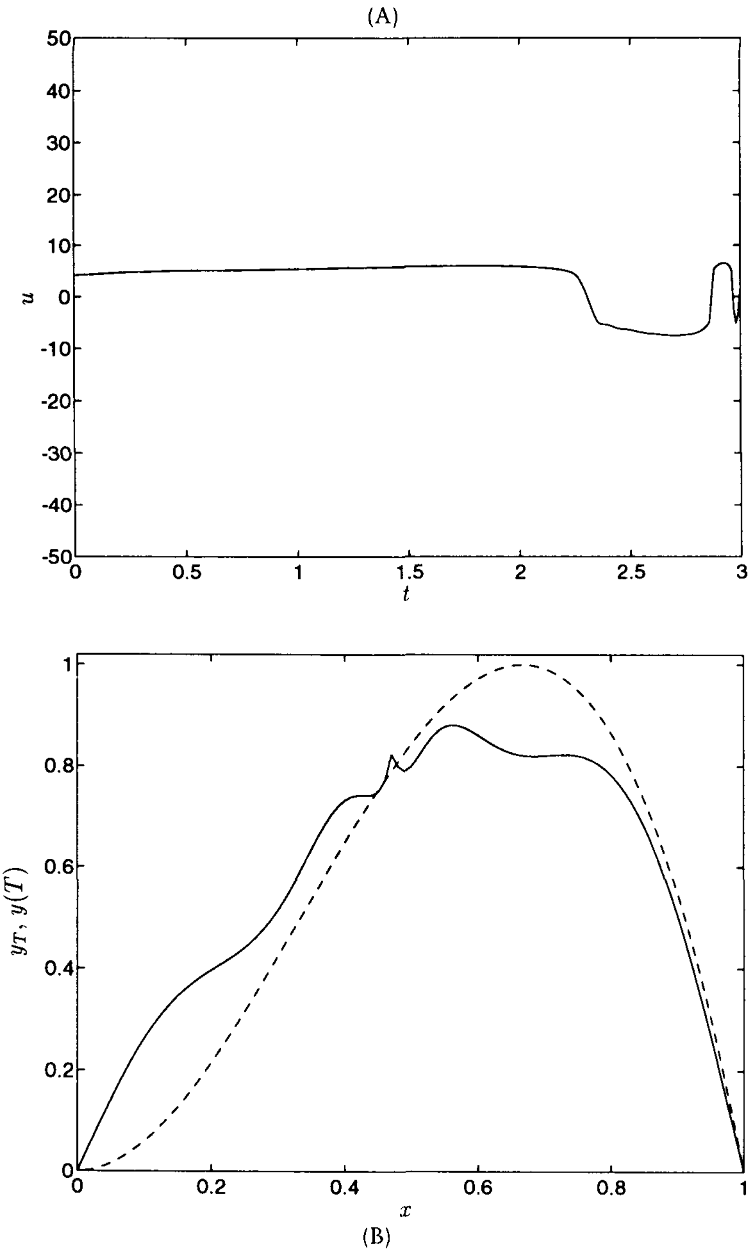
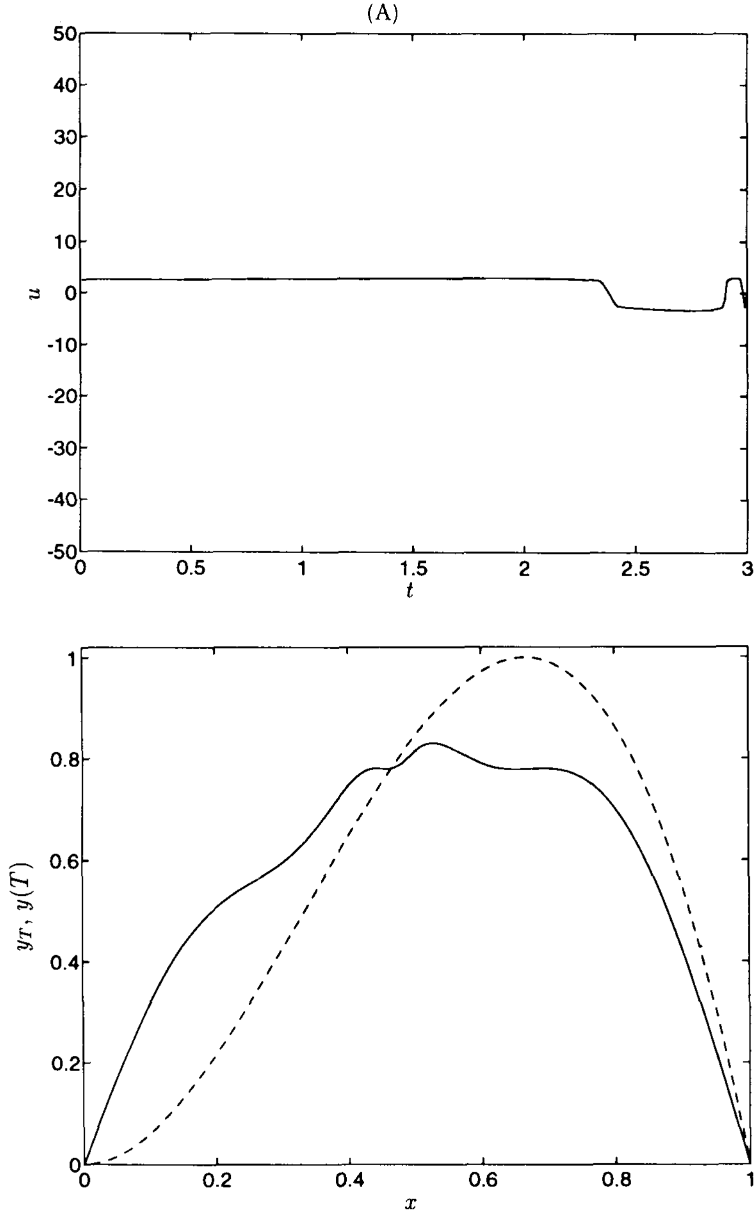


Fig. 18. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466): $s = 6$, $T = 3$, $b = \sqrt{2}/3$, $k = 10^7$, $h = \Delta t = 10^{-2}$).

(A)



(B)

Fig. 19. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466): $s = 10$, $T = 3$, $b = \sqrt{2}/3$, $k = 10^7$, $h = \Delta t = 10^{-2}$).

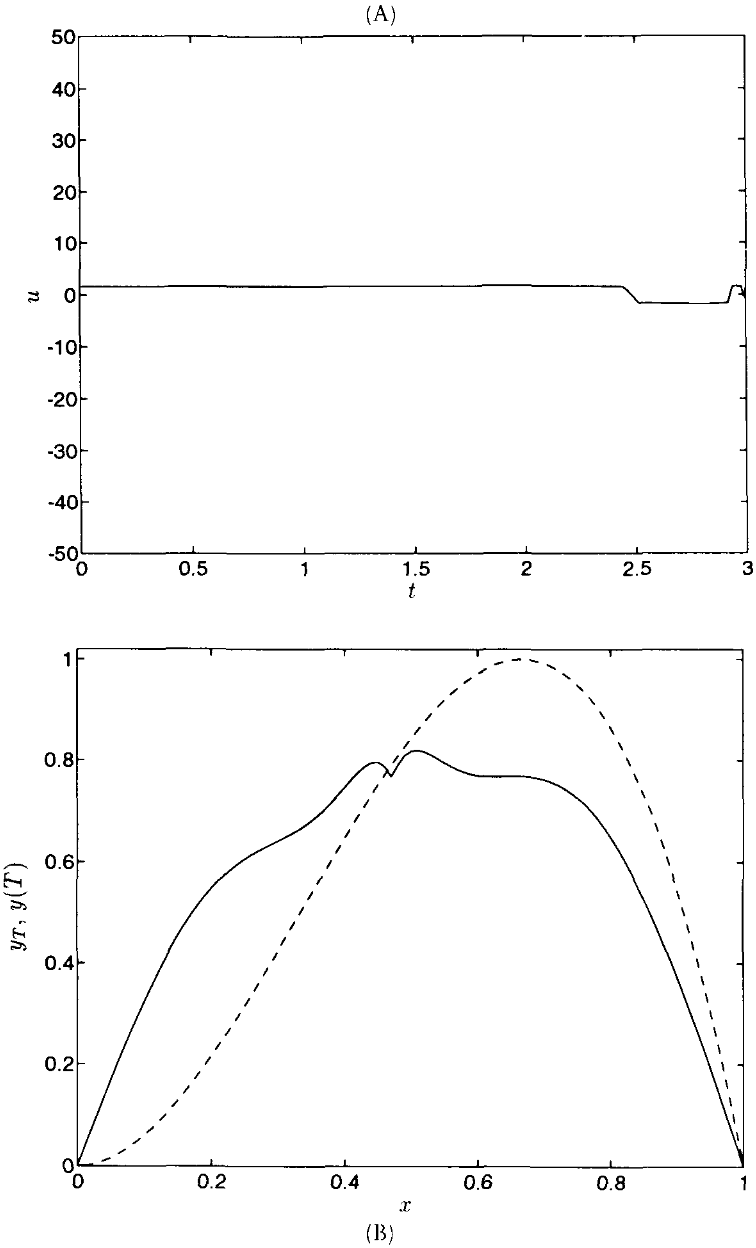
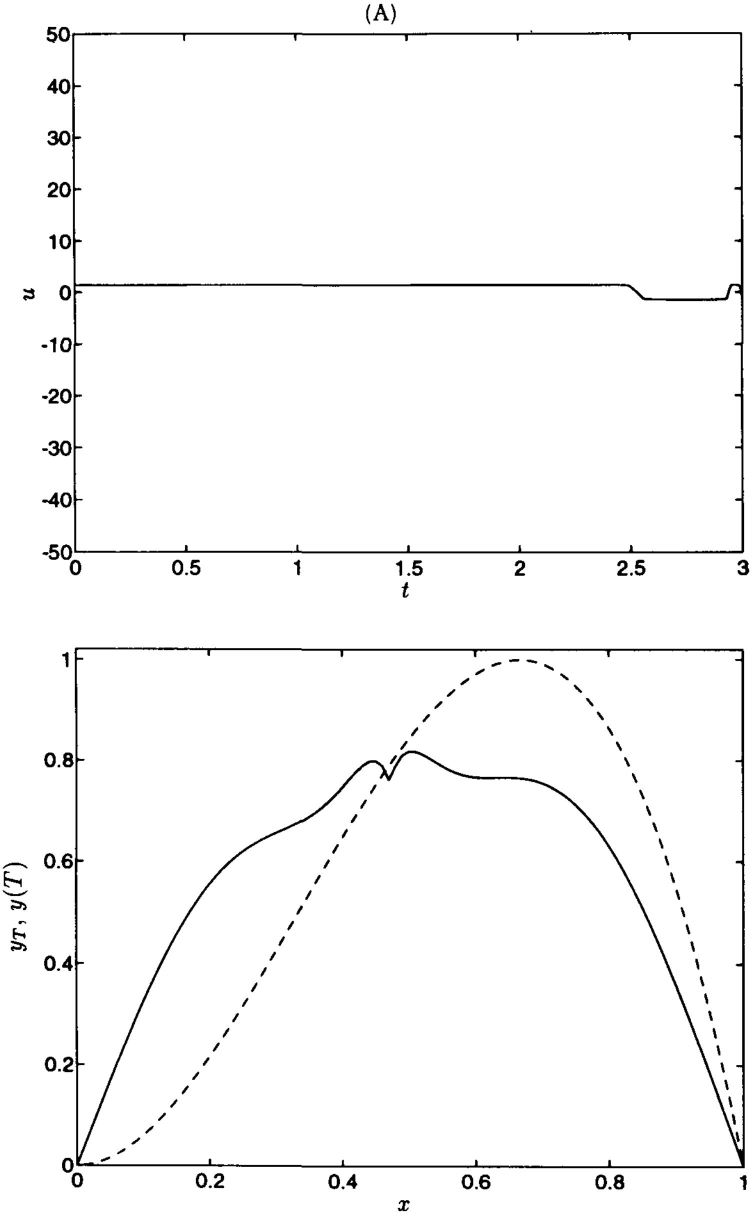


Fig. 20. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466): $s = 20$, $T = 3$, $b = \sqrt{2}/3$, $k = 10^7$, $h = \Delta t = 10^{-2}$).

(A)



(B)

Fig. 21. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466): $s = 30$, $T = 3$, $b = \sqrt{2}/3$, $k = 10^7$, $h = \Delta t = 10^{-2}$).

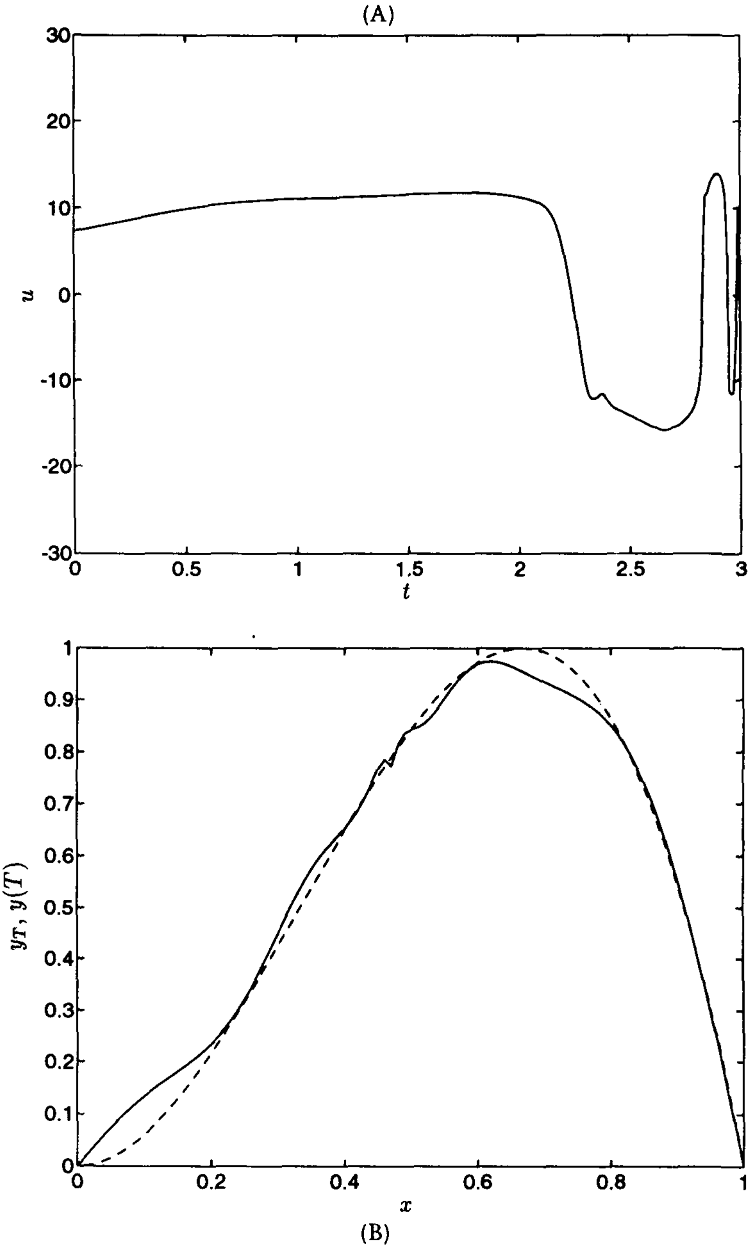


Fig. 22. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466): $s = 6$, $T = 3$, $b = \sqrt{2}/3$, $k = 2 \times 10^9$, $h = \Delta t = 10^{-2}$).

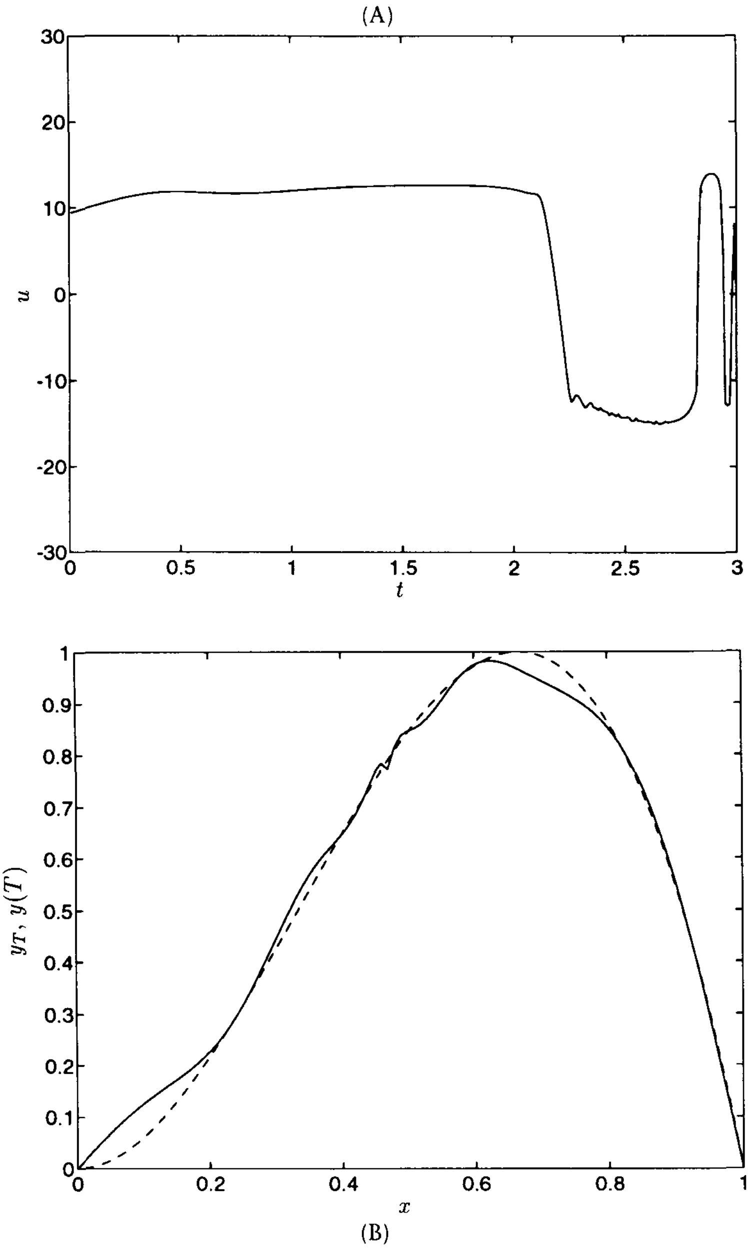
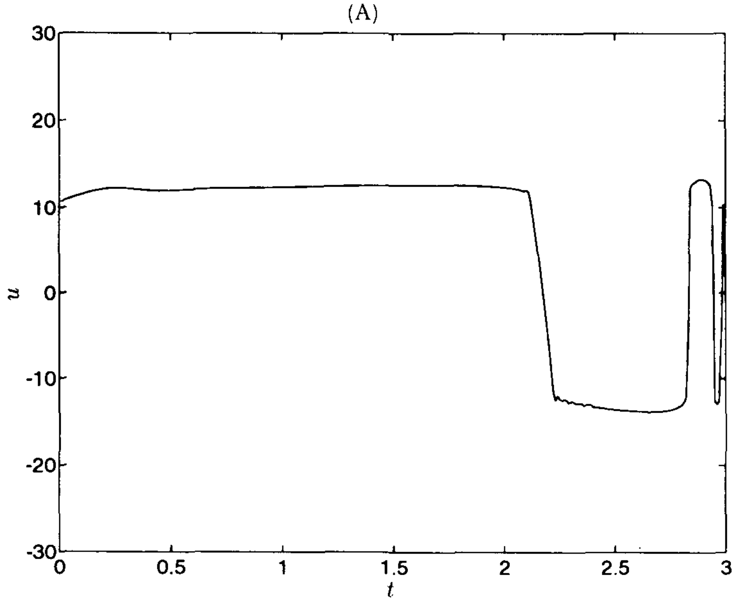


Fig. 23. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466): $s = 6$, $T = 3$, $b = \sqrt{2}/3$, $k = 2 \times 10^9$, $h = \Delta t = 10^{-2}$).



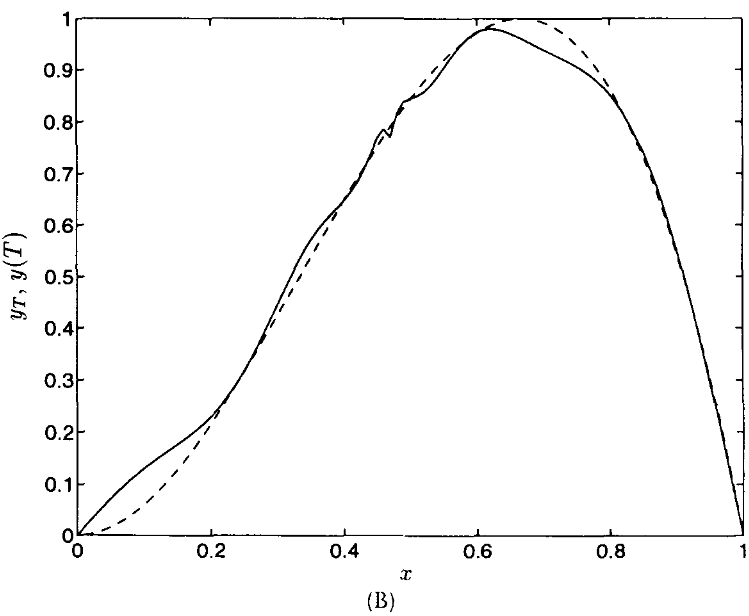


Fig. 24. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466): $s = 20, T = 3, b = \sqrt{2}/3, k = 10^{25}, h = \Delta t = 10^{-2}$).

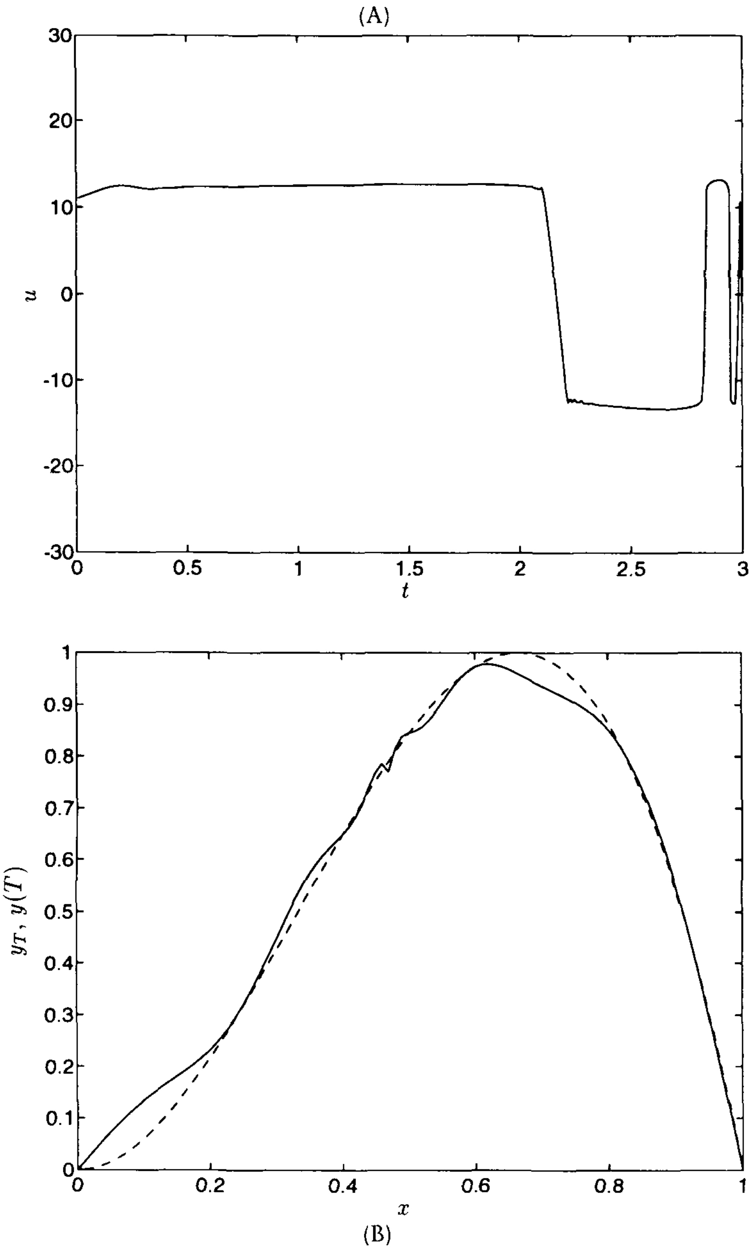


Fig. 25. (a) Variation of the optimal control and (b) comparison between y_T and $y^*(T)$ (target function (1.466): $s = 30, T = 3, b = \sqrt{2}/3, k = 10^{30}, h = \Delta t = 10^{-2}$).

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